

## ARTIN-SCHELTER REGULAR ALGEBRAS AND CATEGORIES

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ABSTRACT. Motivated by constructions in the representation theory of finite dimensional algebras we generalize the notion of Artin-Schelter regular algebras of dimension  $n$  to algebras and categories to include Auslander algebras and a graded analogue for infinite representation type. A generalized Artin-Schelter regular algebra or a category of dimension  $n$  is shown to have common properties with the classical Artin-Schelter regular algebras. In particular, when they admit a duality, then they satisfy Serre duality formulas and the Ext-category of nice sets of simple objects of maximal projective dimension  $n$  is a finite length Frobenius category.

## INTRODUCTION

Artin-Schelter regular rings have been introduced as non-commutative analogues of polynomial rings by Artin and Schelter in [2]. This has become the starting point of a rich theory of non-commutative algebra and algebraic geometry. Our initial source of inspiration and examples comes from the representation theory of Artin/finite dimensional algebras. A finite dimensional algebra  $\Lambda$  has either a finite or an infinite number of isomorphism classes of indecomposable finitely generated modules, so-called finite or infinite type, respectively. In either case, one associates the Auslander algebra/category  $\mathfrak{A}_\Lambda$  to the algebra  $\Lambda$ , and we have observed that  $\mathfrak{A}_\Lambda$  behaves very much like an Artin-Schelter regular algebra, namely

- (i) the global dimension is 2 (finite) ([4]),
- (ii) all the simples of maximal projective dimension satisfy the Gorenstein condition, or equivalently the 2-simple condition ([14]),
- (iii) the Ext-algebra/category of (nice) sets of simples of maximal projective dimension is a finite dimensional/length Frobenius algebra/category.

This motivated us to investigate this in further detail and generality, and the purpose of this paper is to present a generalization of Artin-Schelter regular algebras to algebras and categories that include the Auslander algebras/categories occurring in representation theory of Artin/finite dimensional algebras. Furthermore, as pointed out to us by Osamu Iyama, the additive closure of a  $n$ -cluster tilting

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module over a finite dimensional algebra is a generalized Artin-Schelter regular category. For rings the same generalization or class of rings has been considered by Iyama in [14], where Noetherian rings having finite global dimension  $n$  satisfying the  $n$ -simple condition have characterized in terms of properties of the minimal injective resolution of the ring and the opposite ring. Here the main motivation was to study orders over complete discrete valuation rings.

No assumption on the Gelfand-Kirillov dimension is needed in Iyama's work. Also in our definition of a generalized Artin-Schelter regular algebra or category, we impose no requirement on having finite Gelfand-Kirillov dimension. From our point of view this does not enter or effect the definition, and in addition we construct examples of generalized Artin-Schelter regular algebras which have infinite Gelfand-Kirillov dimension. But an algebra in the class of examples we construct has finite Gelfand-Kirillov dimension if and only if it is Noetherian. This last question is, to our knowledge, an open problem for Artin-Schelter regular algebras.

In this paper we freely use the results from [20]. Furthermore, in a forthcoming paper we use the results in this paper to show that the some properties of the associated graded Auslander category of a component in the Auslander-Reiten quiver of a finite dimensional algebra is reflected as Noetherianity and different Gelfand-Kirillov dimensions.

Recall that a graded connected  $K$ -algebra  $\Lambda$  over a field  $K$  is *Artin-Schelter regular of dimension  $n$* , if  $\text{gldim } \Lambda = n < \infty$ , it has finite Gelfand-Kirillov dimension, and the unique simple graded  $\Lambda$ -module  $S$  satisfies the Gorenstein condition, that is,  $\text{Ext}_\Lambda^n(S, \Lambda) = (0)$  for  $i \neq n$  and  $\text{Ext}_\Lambda^n(S, \Lambda)$  is isomorphic to some graded shift of the unique simple graded  $\Lambda^{\text{op}}$ -module. As mentioned above our goal is to extend this notion of Artin-Schelter regular algebras such that it includes the (graded) Auslander algebras and categories. We do this in two steps. In Section 1 we first consider the class of rings considered by Iyama in [14] (even semiperfect) and in the graded setting non-connected algebras. Properties of generalized Artin-Schelter regular algebras are discussed in Section 2, and in particular we show that the Ext-algebra of a finite set of simple modules of maximal projective dimension permuted the functor  $D \text{Ext}_\Lambda^n(-, \Lambda)$  is a finite dimensional Frobenius algebra. Section 3 is devoted to reviewing the polynomial ring  $K[x_1, x_2, \dots, x_n]$  as an example of ungraded generalized Artin-Schelter algebras to illustrate our results. We construct in Section 4 a family of generalized Artin-Schelter regular algebras including some non-Noetherian and with infinite Gelfand-Kirillov dimension. This is contrary what is believed to be true for classical Artin-Schelter regular algebras, but we show that they are Noetherian if and only if they have finite Gelfand-Kirillov dimension. We take the second step in Section 5 and give the definition of generalized Artin-Schelter regular  $K$ -categories. In Section 6, we show that our generalized Artin-Schelter regular categories share similar properties with the generalized Artin-Schelter regular algebras. The final section, Section 7, is devoted to studying generalized Artin-Schelter regular categories  $\mathcal{C}$  of global dimension 2, where we show that  $\mathcal{C}$  is coherent and  $\text{Gr}(\mathcal{C})$  is abelian when  $\mathcal{C}$  is generated in degrees 0 and 1.

## 1. GENERALIZED ARTIN-SCHELTER REGULAR ALGEBRAS

This section is devoted to generalizing the classical notion of Artin-Schelter regular algebras of Artin and Schelter from [2]. For ungraded rings the same class of Noetherian rings has been considered by Iyama in [14]. We have seen the need of

such a generalization through introducing a Koszul theory associated to any finite dimensional algebra in [20] (and applications in forthcoming papers). The results in this section serve as a model for our generalization to categories in Section 5.

We start the section by reviewing how the Koszul theory/object we associate to any finite dimensional algebra in [20] is similar in behavior to Artin-Schelter regular algebras. Then we develop the necessary theory to end the section with our definition of a generalized Artin-Schelter regular algebra.

Before indulging in the gruesome details of our generalization of Artin-Schelter regular algebras, we first try to motivate the need for such a generalization. For another motivation we refer the reader to the papers [12, 13, 14]. First we consider the class of Auslander algebras. They correspond to the finite dimensional algebras (Artin algebras) with only a finite number of isomorphism classes of indecomposable finitely generated left modules (finite representation type). They are defined as finite dimensional algebras with dominant dimension at least 2 and global dimension at most 2. Let  $\Lambda$  be a finite dimensional  $K$ -algebra of finite representation type, and denote by  $M$  the direct sum of all indecomposable finitely generated left modules with one from each isomorphism class. Then  $\Gamma = \text{End}_\Lambda(M)^{\text{op}}$  is the corresponding Auslander algebra. The simple  $\Gamma$ -modules are in one-to-one correspondence with the indecomposable  $\Lambda$ -modules. Let  $\{S_C\}_{C \in \text{ind } \Lambda}$  be the simple  $\Gamma$ -modules, where  $\text{ind } \Lambda$  is the set of isomorphism classes of indecomposable finitely generated left  $\Lambda$ -modules. A minimal projective resolution of  $S_C$  is given by

$$0 \rightarrow \text{Hom}_\Lambda(M, \tau C) \rightarrow \text{Hom}_\Lambda(M, B) \rightarrow \text{Hom}_\Lambda(M, C) \rightarrow S_C \rightarrow 0$$

if  $C$  is non-projective and  $0 \rightarrow \tau C \rightarrow B \rightarrow C \rightarrow 0$  is the almost split sequence ending in  $C$ , and by

$$0 \rightarrow \text{Hom}_\Lambda(M, \tau P) \rightarrow \text{Hom}_\Lambda(M, P) \rightarrow S_P \rightarrow 0$$

if  $C = P$  is projective. Here  $\tau$  denotes the Jacobson radical of  $\Lambda$ . For the definition and further properties of almost split sequences we refer to [6]. It is easy to see that all the simple modules corresponding to non-projective indecomposable modules satisfy the Gorenstein condition or the 2-simple condition, that is,  $\text{Ext}_\Gamma^i(S_C, \Gamma) = (0)$  for  $i \neq 2$  and  $\text{Ext}_\Gamma^2(S_C, \Gamma)$  is a simple  $\Gamma^{\text{op}}$ -module. Note also that they have the highest possible projective dimension 2, as the global dimension of  $\Gamma$  is 2. Hence, in this case, all the simple modules of highest possible projective dimension behave as over an Artin-Schelter regular algebra. This is called the  $n$ -simple condition in [14]. Similarly one can consider the replacement of the Auslander algebra for a finite dimensional algebra of infinite representation type, namely, the category of additive functors from  $(\text{mod } \Lambda)^{\text{op}}$  to  $\text{Mod } K$ . Here one can find similar behavior. In these situations we are able to prove similar results as for the classical Artin-Schelter regular algebras. This motivates our generalization of Artin-Schelter regular algebras and also Artin-Schelter regular  $K$ -categories.

To exemplify what we mean by similar results we give a concrete example.

**Example 1.1.** Let  $\Sigma = K[x]/(x^3)$  for a field  $K$ . Then  $\Sigma$  has three indecomposable finitely generated modules,  $U = K[x]/(x)$ ,  $L = K[x]/(x^2)$  and  $\Sigma$ . Let  $M = U \amalg L \amalg \Sigma$ , then  $\Lambda = \text{End}_\Sigma(M)^{\text{op}}$  is the corresponding Auslander algebra for  $\Sigma$ . The algebra  $\Lambda$  is a finite dimensional algebra over  $K$  with three simple modules  $S_U$ ,  $S_L$  and  $S_\Sigma$ . The exact sequences  $0 \rightarrow U \rightarrow L \rightarrow U \rightarrow 0$ ,  $0 \rightarrow L \rightarrow \Sigma \amalg U \rightarrow L \rightarrow 0$

and  $0 \rightarrow L \rightarrow \Sigma$  induce the following projective resolutions

$$0 \rightarrow \text{Hom}_\Sigma(M, U) \rightarrow \text{Hom}_\Sigma(M, L) \rightarrow \text{Hom}_\Sigma(M, U) \rightarrow S_U \rightarrow 0,$$

$$0 \rightarrow \text{Hom}_\Sigma(M, L) \rightarrow \text{Hom}_\Sigma(M, \Sigma \amalg U) \rightarrow \text{Hom}_\Sigma(M, L) \rightarrow S_L \rightarrow 0,$$

and

$$0 \rightarrow \text{Hom}_\Sigma(M, L) \rightarrow \text{Hom}_\Sigma(M, \Sigma) \rightarrow S_\Sigma \rightarrow 0$$

of the three simple  $\Lambda$ -modules  $S_U$ ,  $S_L$  and  $S_\Sigma$ . By applying the functor  $\text{Hom}_\Lambda(-, \Lambda)$  to the first two sequences, it follows easily that  $\text{Ext}_\Lambda^i(S, \Lambda) = (0)$  for  $i \neq 2$ , and that  $\text{Ext}_\Lambda^2(S, \Lambda) \simeq S^{\text{op}}$  for  $S$  equal to  $S_U$  or  $S_L$ . Hence the simples  $S_U$  and  $S_L$  satisfies the 2-simple condition, while  $S_\Sigma$  does not. Now let  $T = S_U \amalg S_L$ , and consider  $\Gamma = \text{Ext}_\Lambda^*(T, T)$ . It is easy to see that  $\Gamma$  is isomorphic to the path algebra

$$K \left( 1 \begin{array}{c} \xrightarrow{\alpha} \\ \xleftarrow{\beta} \end{array} 2 \right) / (\alpha\beta\alpha, \beta\alpha\beta), \text{ which is Frobenius. Recall that for an Artin-Schelter}$$

regular algebra the Ext-algebra of the simple graded module is Frobenius. Hence the above example illustrate a similar behavior, and for our generalized Artin-Schelter regular algebras of dimension  $n$  we show that the Ext-algebra of certain subsets of simple modules satisfying the  $n$ -simple condition (also over the opposite algebra) is finite dimensional Frobenius.

Now we start discussing the generalization of the notion of Artin-Schelter regular algebras. Let  $\Lambda$  be any ring and  $n$  a positive integer. We denote by  $\mathcal{H}_\Lambda^n$  the full subcategory of all left  $\Lambda$ -modules  $M$  with projective dimension  $n$  such that  $\text{Ext}_\Lambda^i(M, \Lambda) = (0)$  for  $i \neq n$  and with a projective resolution consisting of finitely generated  $\Lambda$ -modules. Similar subcategories of  $\Lambda$ -modules are considered in [14], and similar results as below are proven. For completeness we include the proofs here. Let  $\text{tr}_\Lambda^n = \text{Ext}_\Lambda^n(-, \Lambda): \text{Mod } \Lambda \rightarrow \text{Mod } \Lambda^{\text{op}}$ . Denote by  $(-)^*$  the functor  $\text{Hom}_\Lambda(-, \Lambda): \text{Mod } \Lambda \rightarrow \text{Mod } \Lambda^{\text{op}}$ .

**Proposition 1.2.** (a) *The subcategory  $\mathcal{H}_\Lambda^n$  is closed under extensions. If  $\text{gldim } \Lambda = n$ , then  $\mathcal{H}_\Lambda^n$  is closed under cokernels of monomorphisms.*  
 (b) *The functor  $\text{tr}_\Lambda^n: \text{Mod } \Lambda \rightarrow \text{Mod } \Lambda^{\text{op}}$  restricts to a functor  $\text{tr}_\Lambda^n: \mathcal{H}_\Lambda^n \rightarrow \mathcal{H}_{\Lambda^{\text{op}}}^n$ , and here it is an exact duality.*  
 (c) *For any  $\Lambda$ -module  $M$  in  $\mathcal{H}_\Lambda^n$  and any  $\Lambda$ -module  $L$  there are isomorphisms*

$$\text{Tor}_i^\Lambda(\text{tr}_\Lambda^n(M), L) \simeq \text{Ext}_\Lambda^{n-i}(M, L)$$

*for all  $i \geq 0$ .*

*Proof.* Let  $\eta: 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  be an exact sequence in  $\text{Mod } \Lambda$ .

(a) If  $A$  and  $C$  are in  $\mathcal{H}_\Lambda^n$ , then the long exact sequence induced by  $\eta$  and the Horseshoe Lemma imply that  $B$  is in  $\mathcal{H}_\Lambda^n$ . Hence  $\mathcal{H}_\Lambda^n$  is closed under extensions.

Suppose that  $\text{gldim } \Lambda = n$ . In the exact sequence  $\eta$  assume that  $A$  and  $B$  are in  $\mathcal{H}_\Lambda^n$ . Then  $C$  also has finite projective dimension. Using the Horseshoe Lemma on the exact sequence  $0 \rightarrow \Omega_\Lambda^1(B) \rightarrow \Omega_\Lambda^1(C) \amalg P \rightarrow A \rightarrow 0$  for some projective  $P$ , imply that  $C$  has a finitely generated projective resolution. The long exact sequence induced by  $\eta$  then imply that  $C$  is in  $\mathcal{H}_\Lambda^n$ .

(b) Let  $M$  be in  $\mathcal{H}_\Lambda^n$ . Let

$$0 \rightarrow P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

be a finitely generated projective resolution of  $M$ . Since  $M$  is in  $\mathcal{H}_\Lambda^n$ , the above long exact sequence gives rise to the long exact sequence

$$0 \rightarrow P_0^* \rightarrow P_1^* \rightarrow \cdots \rightarrow P_{n-1}^* \rightarrow P_n^* \rightarrow \text{Ext}_\Lambda^n(M, \Lambda) \rightarrow 0$$

This shows that  $\text{tr}_\Lambda^n(M)$  has a finitely generated projective resolution of length  $n$ . Applying the functor  $\text{Hom}_{\Lambda^{\text{op}}}(-, \Lambda)$ , which we also denote by  $(-)^*$ , we obtain the following commutative diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & P_n & \longrightarrow & P_{n-1} & \longrightarrow & \cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (\text{tr}_\Lambda^n(M))^* & \longrightarrow & P_n^{**} & \longrightarrow & P_{n-1}^{**} & \longrightarrow & \cdots & \longrightarrow & P_1^{**} & \longrightarrow & P_0^{**} & \longrightarrow & \text{tr}_{\Lambda^{\text{op}}}^n(\text{tr}_\Lambda^n(M)) \longrightarrow 0 \end{array}$$

where all vertical maps  $P_i \rightarrow P_i^{**}$  are isomorphisms. In addition,  $P_1^{**} \rightarrow P_0^{**} \rightarrow \text{tr}_{\Lambda^{\text{op}}}^n(\text{tr}_\Lambda^n(M)) \rightarrow 0$  is exact. It follows from this that  $\text{Ext}_{\Lambda^{\text{op}}}^i(\text{tr}_\Lambda^n(M), \Lambda) = (0)$  for all  $i \neq n$ . Hence  $\text{tr}_\Lambda^n(M)$  is in  $\mathcal{H}_{\Lambda^{\text{op}}}^n$  and  $M \simeq \text{tr}_{\Lambda^{\text{op}}}^n(\text{tr}_\Lambda^n(M))$ . This shows that  $\text{tr}_\Lambda^n: \mathcal{H}_\Lambda^n \rightarrow \mathcal{H}_{\Lambda^{\text{op}}}^n$  is a duality.

(c) Let  $M$  be in  $\mathcal{H}_\Lambda^n$  with projective resolution as above in (b). Then the complexes  $0 \rightarrow P_0^* \otimes_\Lambda L \rightarrow P_1^* \otimes_\Lambda L \rightarrow \cdots \rightarrow P_{n-1}^* \otimes_\Lambda L \rightarrow P_n^* \otimes_\Lambda L \rightarrow 0$  and  $0 \rightarrow \text{Hom}_\Lambda(P_0, L) \rightarrow \text{Hom}_\Lambda(P_1, L) \rightarrow \cdots \rightarrow \text{Hom}_\Lambda(P_{n-1}, L) \rightarrow \text{Hom}_\Lambda(P_n, L) \rightarrow 0$  are isomorphic. The claims follow directly from this.  $\square$

If a simple module  $S$  is in  $\mathcal{H}_{\Lambda^{\text{op}}}^n$ , it is not clear that  $\text{tr}_{\Lambda^{\text{op}}}^n(S)$  in  $\mathcal{H}_\Lambda^n$ , again is a simple module. For Noetherian rings this property is studied in [14]. There a Noetherian ring  $\Lambda$  is said to *satisfy the  $n$ -simple condition* if, for every simple  $\Lambda$ -module  $S$  with  $\text{pd}_\Lambda S = n$ , we have that  $\text{Ext}_\Lambda^i(S, \Lambda) = (0)$  for  $0 \leq i < n$  and  $\text{Ext}_\Lambda^n(S, \Lambda)$  is a simple  $\Lambda^{\text{op}}$ -module. The following follows directly from [14, Proposition 6.3].

**Theorem 1.3** ([14]). *Let  $\Lambda$  be a Noetherian ring with  $\text{gldim } \Lambda \leq n$ . Then  $\Lambda$  and  $\Lambda^{\text{op}}$  satisfy the  $n$ -simple condition if and only if, in a minimal injective resolution*

$$0 \rightarrow \Lambda \rightarrow I^0 \rightarrow I^1 \rightarrow I^2 \rightarrow \cdots$$

*we have  $\text{fd}_\Lambda I^i < n$  for  $0 \leq i < n$  and the same property for  $\Lambda^{\text{op}}$ .*

We do not know if there is a similar characterization of the  $n$ -simple condition for non-Noetherian rings. It is shown in [15, The proof of Theorem 2.10] that  $\text{Ext}_\Lambda^n(S, \Lambda)$  has finite length over  $\Lambda^{\text{op}}$ , whenever  $S$  is in  $\mathcal{H}_\Lambda^n$ . So there is always simple submodules of  $\text{Ext}_\Lambda^n(S, \Lambda)$  for  $S$  in  $\mathcal{H}_\Lambda^n$ . We show next that, if one of these simples are in  $\mathcal{H}_{\Lambda^{\text{op}}}^n$ , then this is enough for having the  $n$ -simple condition.

**Lemma 1.4.** *Let  $\Lambda$  be a ring of finite global dimension  $n$ , and let  $S$  be a simple module in  $\mathcal{H}_{\Lambda^{\text{op}}}^n$ . Assume that  $\text{tr}_{\Lambda^{\text{op}}}^n(S)$  contains a simple module in  $\mathcal{H}_\Lambda^n$  as a submodule. Then  $\text{tr}_{\Lambda^{\text{op}}}^n(S)$  is a simple module (and it is in  $\mathcal{H}_\Lambda^n$ ).*

*Proof.* Let  $S$  be a simple module in  $\mathcal{H}_{\Lambda^{\text{op}}}^n$ . Assume that  $T \subseteq \text{tr}_{\Lambda^{\text{op}}}^n(S)$  is a simple submodule with  $T$  in  $\mathcal{H}_\Lambda^n$ . By Proposition 1.2 we obtain an exact sequence

$$0 \rightarrow \text{tr}_\Lambda^n(\text{tr}_{\Lambda^{\text{op}}}^n(S)/T) \rightarrow \text{tr}_\Lambda^n(\text{tr}_{\Lambda^{\text{op}}}^n(S)) \rightarrow \text{tr}_\Lambda^n(T) \rightarrow 0$$

in  $\mathcal{H}_\Lambda^n$ , where  $\text{tr}_\Lambda^n(\text{tr}_{\Lambda^{\text{op}}}^n(S)) \simeq S$  and  $\text{tr}_\Lambda^n(T)$  is nonzero. It follows that  $S \simeq \text{tr}_\Lambda^n(T)$  and  $\text{tr}_{\Lambda^{\text{op}}}^n(S) \simeq T$ .  $\square$

Let  $\mathcal{S}^n$  denote the simple  $\Lambda$ -modules which are in  $\mathcal{H}_\Lambda^n$ . Then  $\mathcal{H}_\Lambda^n$  contains all  $\Lambda$ -modules of finite length with composition factors only in  $\mathcal{S}^n$  by Proposition 1.2. Our next aim is to find sufficient conditions on  $\Lambda$  such that  $\mathcal{H}_\Lambda^n$  is exactly the full category of  $\Lambda$ -modules of finite length with composition factors only in  $\mathcal{S}^n$ .

Define a subfunctor  $t^n: \text{Mod } \Lambda \rightarrow \text{Mod } \Lambda$  of the identity as follows. Let  $M$  be a  $\Lambda$ -module. Then let  $t^n(M) = \sum_{L \subseteq M} L$ , where the sum is taken over all submodules  $L$  of  $M$ , where  $L$  has finite length with composition factors only in  $\mathcal{S}^n$ . We say that a module  $M$  is  $\mathcal{H}_\Lambda^n$ -torsion if  $M = t^n(M)$ . A module  $M$  is  $\mathcal{H}_\Lambda^n$ -torsion free if  $t^n(M) = (0)$ . Next we show that  $t^n$  is a radical in  $\text{Mod } \Lambda$  for all  $n \geq 1$ .

**Lemma 1.5.** *Let  $\Lambda$  be a semiperfect left Noetherian ring, or let  $\Lambda$  be a positively graded ring such that  $\Lambda_0 \simeq K^t$  for some  $t$ ,  $\dim_K \Lambda_i < \infty$  for all  $i \geq 0$ , and  $\Lambda_{\geq 1}$  is a finitely generated left  $\Lambda$ -module. Then  $t^n(M/t^n(M)) = (0)$  for any  $\Lambda$ -module or graded  $\Lambda$ -module  $M$ , or equivalently  $M/t^n(M)$  is  $\mathcal{H}_\Lambda^n$ -torsion free.*

*Proof.* Let  $M$  be a  $\Lambda$ -module. Assume that  $M/t^n(M)$  has a submodule of finite length with compositions factors only in  $\mathcal{S}^n$ . Then  $M/t^n(M)$  has a simple submodule  $S$  from  $\mathcal{S}^n$ , where  $S = \Lambda/\mathfrak{m}$  for some maximal left (graded) ideal  $\mathfrak{m}$ . Let  $x$  in  $M$  be such that  $\bar{x} = 1 + \mathfrak{m}$ . Then  $\mathfrak{m}x$  is contained in  $t^n(M)$ . If  $\Lambda$  is a left Noetherian ring, then  $\mathfrak{m}$  is finitely generated. If  $\Lambda$  is graded as above, then  $\mathfrak{m} = \mathfrak{m}_0 + \Lambda_{\geq 1}$  for some maximal ideal  $\mathfrak{m}_0$  in  $\Lambda_0$ . Hence, also here  $\mathfrak{m}$  is finitely generated under the assumptions in the graded case. In either case, it follows that  $\mathfrak{m}x$  is contained in a submodule  $L$  of  $M$  of finite length with composition factors only in  $\mathcal{S}^n$ . Furthermore, there is an exact sequence  $0 \rightarrow \mathfrak{m}x \rightarrow \Lambda x \rightarrow S \rightarrow 0$ , so that  $\Lambda x$  is submodule of  $M$  of finite length with composition factors only in  $\mathcal{S}^n$ . Hence  $x$  is in  $t^n(M)$ . This is a contradiction, so we infer that  $t^n(M/t^n(M)) = (0)$ .  $\square$

In some settings torsion free is the same as a submodule of a free module. If, in addition, the ring has finite global dimension, then the projective dimension of a torsion free module is always at most one less than the global dimension. We show next that the behavior of  $\mathcal{H}_\Lambda^n$ -torsion free modules are similar when  $\Lambda$  satisfies some additional properties compared to Lemma 1.5.

**Proposition 1.6.** *Let  $\Lambda$  be a semiperfect left Noetherian ring, or let  $\Lambda$  be a positively graded ring such that  $\Lambda_0 \simeq K^t$  for some  $t$ ,  $\dim_K \Lambda_i < \infty$  for all  $i \geq 0$ , and  $\Lambda_{\geq 1}$  is a finitely generated left  $\Lambda$ -module.*

*Assume that  $\Lambda$  has finite (graded) global dimension  $n$ , that the category  $\mathcal{H}_{\Lambda^{\text{op}}}^n$  contains all (graded) simple  $\Lambda^{\text{op}}$ -modules of maximal projective dimension  $n$  and that  $\text{tr}_{\Lambda^{\text{op}}}^n(S)$  has a simple module from  $\mathcal{H}_\Lambda^n$  as a submodule for all (graded) simple  $\Lambda^{\text{op}}$ -modules  $S$  of maximal projective dimension  $n$ . Then any finitely generated  $\mathcal{H}_\Lambda^n$ -torsion free  $\Lambda$ -module has projective dimension at most  $n - 1$ .*

*Proof.* Let  $\mathcal{S}_{\Lambda^{\text{op}}}^n$  be all the (graded) simple  $\Lambda^{\text{op}}$ -modules of maximal projective dimension  $n$ , and assume that  $\mathcal{S}_{\Lambda^{\text{op}}}^n$  is in  $\mathcal{H}_{\Lambda^{\text{op}}}^n$ . Let  $S' = \text{tr}_{\Lambda^{\text{op}}}^n(S)$  for a simple  $\Lambda^{\text{op}}$ -module  $S$  in  $\mathcal{S}_{\Lambda^{\text{op}}}^n$ . By Proposition 1.2 (c) we have that  $\text{Ext}_\Lambda^{n-i}(S', L) \simeq \text{Tor}_i^\Lambda(S, L)$  for all  $i$  and for any  $\Lambda$ -module  $L$ . In particular, assuming that  $L$  is  $\mathcal{H}_\Lambda^n$ -torsion free and by Lemma 1.4 the module  $S'$  is a simple module in  $\mathcal{H}_\Lambda^n$ , we obtain

$$(0) = \text{Hom}_\Lambda(S', L) \simeq \text{Tor}_n^\Lambda(S, L) \simeq \text{Tor}_1^\Lambda(S, \Omega_\Lambda^{n-1}(L))$$

for simple  $\Lambda^{\text{op}}$ -modules  $S$  in  $\mathcal{S}_{\Lambda^{\text{op}}}^n$ . For any other simple  $\Lambda^{\text{op}}$ -module  $S''$  the projective dimension of  $S''$  is at most  $n - 1$ , so that  $\text{Tor}_1^\Lambda(S'', \Omega_\Lambda^{n-1}(L)) = (0)$  also for these simples. Hence  $\text{Tor}_1^\Lambda(S, \Omega_\Lambda^{n-1}(L)) = (0)$  for all simple  $\Lambda^{\text{op}}$ -modules  $S$ .

If  $\Lambda$  is graded as in the claim of the proposition, then  $\Omega_\Lambda^{n-1}(L)$  is a graded  $\Lambda$ -module bounded below. Let  $0 \rightarrow \Omega_\Lambda^n(L) \rightarrow P \rightarrow \Omega_\Lambda^{n-1}(L) \rightarrow 0$  be a graded projective cover. Then  $(0) = \text{Tor}_1^\Lambda(\Lambda/\text{rad } \Lambda, \Omega_\Lambda^{n-1}(L)) \simeq \Omega_\Lambda^n(L)/\text{rad } \Lambda \Omega_\Lambda^n(L)$ , where  $\text{rad } \Lambda$  is the graded Jacobson radical of  $\Lambda$ . Each graded part of  $\Omega_\Lambda^n(L)$  is a finitely generated  $\Lambda_0$ -module. Using Nakayama's Lemma we infer that  $\Omega_\Lambda^n(L) = (0)$ .

If  $\Lambda$  is left Noetherian, then  $\Omega_\Lambda^{n-1}(L)$  is a finitely generated flat  $\Lambda$ -module, hence it is projective. In either case it follows that the projective dimension of  $L$  is at most  $n - 1$ .  $\square$

A  $\mathcal{H}_\Lambda^n$ -torsion module  $M$  need not to be in  $\mathcal{H}_\Lambda^n$  (not even finitely generated), but they share the property that  $\text{Ext}_\Lambda^i(-, \Lambda)$  vanish for all  $i \neq n$  with the modules in  $\mathcal{H}_\Lambda^n$ , as is shown next.

**Lemma 1.7.** *Let  $\Lambda$  be a ring of finite global dimension  $n$  such that all simple  $\Lambda$ -modules of maximal projective dimension  $n$  are in  $\mathcal{H}_\Lambda^n$ . Assume that  $\Lambda$  is a semiperfect left Noetherian ring, or that  $\Lambda$  is a positively graded ring such that  $\Lambda_0 \simeq K^t$  for some  $t$ ,  $\dim_K \Lambda_i < \infty$  for all  $i \geq 0$ , and  $\Lambda_{\geq 1}$  is a finitely generated left  $\Lambda$ -module.*

*Then any  $\mathcal{H}_\Lambda^n$ -torsion module  $M$  satisfies  $\text{Ext}_\Lambda^i(M, \Lambda) = (0)$  for all  $i \neq n$ .*

*Proof.* Let  $M$  be a  $\mathcal{H}_\Lambda^n$ -torsion  $\Lambda$ -module. Then  $M$  can be written as a filtered colimit of submodules  $M_\alpha$  of finite length with composition factors only in  $\mathcal{S}^n$ , such that  $M_\alpha \hookrightarrow M_{\alpha'}$  whenever  $\alpha \leq \alpha'$ . Let  $0 \rightarrow \Lambda \rightarrow I_0 \rightarrow I_1 \rightarrow \cdots \rightarrow I_{n-1} \rightarrow I_n \rightarrow 0$  be a minimal injective resolution of  $\Lambda$ . Since

$$0 \rightarrow (M_\alpha, \Omega_\Lambda^{-j}(\Lambda)) \rightarrow (M_\alpha, I_j) \rightarrow (M_\alpha, \Omega_\Lambda^{-j-1}(\Lambda)) \rightarrow 0$$

induces an exact sequence of inverse system where the first is a surjective system for  $j = 0, 1, \dots, n-2$ , the inverse limit of each of these different systems are exact by the Mittag-Leffler condition. Hence they give rise to the exact sequences

$$0 \rightarrow (M, \Omega_\Lambda^{-j}(\Lambda)) \rightarrow (M, I_j) \rightarrow (M, \Omega_\Lambda^{-j-1}(\Lambda)) \rightarrow 0$$

for  $j = 0, 1, \dots, n-2$ , and therefore  $\text{Ext}_\Lambda^j(M, \Lambda) = (0)$  for all  $j \neq n$ .  $\square$

Now we are ready to give sufficient conditions for  $\mathcal{H}_\Lambda^n$  to be exactly the modules of finite length with composition factors only in  $\mathcal{S}^n$ .

**Proposition 1.8.** *Let  $\Lambda$  be a ring of finite global dimension  $n$ . Assume that  $\Lambda$  is a semiperfect left Noetherian ring, or that  $\Lambda$  is a positively graded ring such that  $\Lambda_0 \simeq K^t$  for some  $t$ ,  $\dim_K \Lambda_i < \infty$  for all  $i \geq 0$ , and  $\Lambda_{\geq 1}$  is a finitely generated left  $\Lambda$ -module. Furthermore, assume that all simple (graded)  $\Lambda$ -modules or  $\Lambda^{\text{op}}$ -module of maximal projective dimension  $n$  are in  $\mathcal{H}_\Lambda^n$  or  $\mathcal{H}_{\Lambda^{\text{op}}}^n$ , respectively, and if  $S$  is a simple (graded)  $\Lambda^{\text{op}}$ -module in  $\mathcal{H}_{\Lambda^{\text{op}}}^n$ , then  $\text{tr}_{\Lambda^{\text{op}}}^n(S)$  has a simple module from  $\mathcal{H}_\Lambda^n$  as a submodule. Then the category  $\mathcal{H}_\Lambda^n$  consists exactly of the modules of finite length with composition factors only in  $\mathcal{S}^n$ .*

*Proof.* By Proposition 1.2 (a) it follows that all modules of finite length with composition factors only in  $\mathcal{S}^n$  are contained in  $\mathcal{H}_\Lambda^n$ .

Conversely, let  $M$  be in  $\mathcal{H}_\Lambda^n$ . By Lemma 1.7 we have that  $\text{Ext}_\Lambda^i(t^n(M), \Lambda) = (0)$  for  $i \neq n$ . The exact sequence  $0 \rightarrow t^n(M) \rightarrow M \rightarrow M/t^n(M) \rightarrow 0$  implies that  $\text{Ext}_\Lambda^i(M/t^n(M), \Lambda) = (0)$  for  $i \neq n$ . Using Proposition 1.6 it follows that  $M/t^n(M) = (0)$ . Therefore  $M = t^n(M)$ , and since  $M$  is finitely generated,  $M$  has finite length and with composition factors only in  $\mathcal{S}^n$ .  $\square$

The above result and the corollary below motivates the following generalization of Artin-Schelter regular algebras.

**Definition 1.9.** Let  $\Lambda$  be a semiperfect two-sided Noetherian ring, or let  $\Lambda$  is a positively graded ring such that  $\Lambda_0 \simeq K^t$  for some  $t$ ,  $\dim_K \Lambda_i < \infty$  for all  $i \geq 0$ , and  $\Lambda_{\geq 1}$  is finitely generated as a left and a right  $\Lambda$ -module. Assume that  $\Lambda$  has finite (graded) global dimension  $n$ .

Then  $\Lambda$  is (graded) *generalized Artin-Schelter regular of dimension  $n$*  if  $\mathcal{H}_\Lambda^n$  and  $\mathcal{H}_{\Lambda^{\text{op}}}^n$  contain all simple (graded)  $\Lambda$ -modules and  $\Lambda^{\text{op}}$ -modules of maximal projective dimension  $n$ , respectively, and  $\text{tr}^n$  maps these simple modules to modules having a simple module from  $\mathcal{H}^n$  on the opposite side as a submodule for all (graded) simple  $\Lambda^{\text{op}}$ -modules  $S$  of maximal projective dimension  $n$ .

We summarize some basic immediate consequences for generalized Artin-Schelter algebras below.

**Corollary 1.10.** *Suppose that  $\Lambda$  is (graded) generalized Artin-Schelter regular of dimension  $n$ . Then the following assertions hold.*

- (a) *The categories  $\mathcal{H}_\Lambda^n$  and  $\mathcal{H}_{\Lambda^{\text{op}}}^n$  consist exactly of the modules of finite length with composition factors only in simple (graded) modules of maximal projective dimension  $n$ .*
- (b) *The functors  $\text{tr}_\Lambda^n: \mathcal{H}_\Lambda^n \rightarrow \mathcal{H}_{\Lambda^{\text{op}}}^n$  and  $\text{tr}_{\Lambda^{\text{op}}}^n: \mathcal{H}_{\Lambda^{\text{op}}}^n \rightarrow \mathcal{H}_\Lambda^n$  are inverse exact dualities, which preserve length, and in particular gives rise to a bijection between the simple (graded)  $\Lambda$ -modules and simple (graded)  $\Lambda^{\text{op}}$ -modules with maximal projective dimension  $n$ .*

*Proof.* The claim in (a) follows directly from Proposition 1.8, and (b) follows from (a) and Proposition 1.2 (b).  $\square$

For a generalized Artin-Schelter regular algebra of dimension  $n$  we can reformulate the definition of the subfunctor  $t^n$  of the identity as follows. Let  $M$  be a  $\Lambda$ -module. Then  $t^n(M) = \sum_{L \subset M} L$ , where the sum is taken over all finite length submodules  $L$  of  $M$  with  $L$  in  $\mathcal{H}_\Lambda^n$ .

Classically a graded connected  $K$ -algebra  $\Lambda$  is Artin-Schelter regular if  $\text{gldim } \Lambda = n < \infty$ , it has finite Gelfand-Kirillov dimension, and the unique simple graded  $\Lambda$ -module  $S$  satisfies the Gorenstein condition, that is,  $\text{Ext}_\Lambda^n(S, \Lambda) = (0)$  for  $i \neq n$  and  $\text{Ext}_\Lambda^n(S, \Lambda)$  is isomorphic to some graded shift of the unique simple graded  $\Lambda^{\text{op}}$ -module. It is then immediate that such a algebra is also Artin-Schelter regular in our sense.

## 2. PROPERTIES OF GENERALIZED ARTIN-SCHELTER REGULAR ALGEBRAS

For classical Artin-Schelter regular algebras one has Serre duality, and there is an intimate relationship with finite dimensional Frobenius algebras. This section investigates analogues of these connections for our generalized Artin-Schelter regular algebras, and we show similar behavior in our case. In particular, if  $\Lambda$  is a (graded) generalized Artin-Schelter regular algebra of dimension  $n$  with duality  $D$ , then

$$D(\text{Ext}_\Lambda^i(M, -)) \simeq \text{Ext}_\Lambda^{n-i}(-, D\text{tr}_\Lambda^n M)$$

for all  $M$  in  $\mathcal{H}_\Lambda^n$  and all  $i$ , and the Ext-algebra of any finite set of simple  $\Lambda$ -modules in  $\mathcal{H}_\Lambda^n$  permuted by  $D\text{tr}_\Lambda^n$  is Frobenius. We show a partial converse of this in Section 6.



We begin by discussing Serre duality type formulas for our generalized Artin-Schelter regular algebras when they admit an ordinary duality.

For generalized Artin-Schelter regular rings  $\Lambda$  of dimension  $n$ , the category  $\mathcal{H}_\Lambda^n$  is exactly the full subcategory of  $\Lambda$ -modules of finite length with composition factors only in  $\mathcal{S}^n$ . Hence,  $\mathcal{H}_\Lambda^n$  and  $\mathcal{H}_{\Lambda^{\text{op}}}^n$  are abelian categories and  $\text{tr}_\Lambda^n : \mathcal{H}_\Lambda^n \rightarrow \mathcal{H}_{\Lambda^{\text{op}}}^n$  is an exact duality. If  $\Lambda$  is semiperfect two-sided Noetherian and a  $K$ -algebra for a field  $K$  and all simple  $\Lambda$ -modules are finite dimensional, then the duality  $D = \text{Hom}_K(-, K)$  induces a duality between the finite length  $\Lambda$ -modules and  $\Lambda^{\text{op}}$ -modules. When  $\Lambda$  is graded as usual with  $\Lambda_0 \simeq K^t$  for some  $t$ , then the duality  $D = \text{Hom}_K(-, K)$  provides such a duality  $D$  as above. When either of the two cases for a generalized Artin-Schelter regular algebra  $\Lambda$  occurs,  $\Lambda$  is said *to have a duality*  $D$ . However it is not true that  $D$  necessarily induces a duality from  $\mathcal{H}_\Lambda^n$  to  $\mathcal{H}_{\Lambda^{\text{op}}}^n$ , unless the categories  $\mathcal{H}^n$  consist for all modules of finite length. In this case we have autoequivalences  $D\text{tr}_\Lambda^n$  and  $\text{tr}_{\Lambda^{\text{op}}}^n D$  of  $\mathcal{H}_\Lambda^n$ . So when giving Serre duality type formulas in the next result, the module  $D\text{tr}_\Lambda^n(M)$  is not necessarily in  $\mathcal{H}_\Lambda^n$  again for  $M$  in  $\mathcal{H}_\Lambda^n$ .

**Proposition 2.1.** *Let  $\Lambda$  be a (graded) generalized Artin-Schelter regular ring of dimension  $n$  having a duality  $D$ . For all pairs of  $\Lambda$ -modules  $L$  and  $M$  with  $M$  in  $\mathcal{H}_\Lambda^n$ , there are natural isomorphisms*

$$\varphi_i : D(\text{Ext}_\Lambda^i(M, L)) \rightarrow \text{Ext}_\Lambda^{n-i}(L, D\text{tr}_\Lambda^n(M))$$

for all  $i$ , and the Auslander-Reiten formulas

$$D(\text{Hom}_\Lambda(M, L)) \simeq \text{Ext}_\Lambda^n(L, D\text{tr}_\Lambda^n(M))$$

and

$$D(\text{Hom}_\Lambda(M, L)) \simeq \text{Ext}_\Lambda^1(L, D\text{Tr}(M)).$$

*Proof.* In either situations we have that  $D(\text{Tor}_i^\Lambda(A, B)) \simeq \text{Ext}_\Lambda^i(B, D(A))$  for a  $\Lambda^{\text{op}}$ -module  $A$ , for a  $\Lambda$ -module  $B$  for all  $i \geq 0$ . The formula

$$\text{Tor}_i^\Lambda(\text{tr}_\Lambda^n(M), L) \simeq \text{Ext}_\Lambda^{n-i}(M, L)$$

for  $M$  in  $\mathcal{H}_\Lambda^n$  from Proposition 1.2 induces the first formula

$$D(\text{Ext}_\Lambda^i(M, L)) \simeq \text{Ext}_\Lambda^{n-i}(L, D\text{tr}_\Lambda^n(M))$$

for all  $i \geq 0$ . The above formula for  $i = 0$  gives the first Auslander-Reiten formula, and using dimension shift the second formula follows directly.  $\square$

Next we show that the Ext-algebra of any finite set of simple  $\Lambda$ -modules in  $\mathcal{H}_\Lambda^n$  permuted by  $D\text{tr}_\Lambda^n$  for a generalized Artin-Schelter regular algebra is Frobenius.

Let  $\Lambda$  be a generalized Artin-Schelter regular ring of dimension  $n$  with a duality  $D$ . In our graded situation  $\Lambda/\mathfrak{r}$  with  $\mathfrak{r}$  being the graded Jacobson radical of  $\Lambda$ , generates the additive closure of all simple graded  $\Lambda$ -modules up to shift. While in the ungraded setting there might be infinitely many simple modules. In both cases denote by  $T$  the direct sum of a finite subset of all simple  $\Lambda$ -modules in  $\mathcal{S}^n$ . Denote by  $\Gamma$  the Yoneda algebra  $\coprod_{i \geq 0} \text{Ext}_\Lambda^i(T, T)$ , and consider the natural functors

$$F = \coprod_{i \geq 0} \text{Ext}_\Lambda^i(-, T) : \text{Mod } \Lambda \rightarrow \text{Gr}(\Gamma)$$

and

$$F' = \coprod_{i \geq 0} \text{Ext}_\Lambda^i(T, -) : \text{Mod } \Lambda \rightarrow \text{Gr}(\Gamma^{\text{op}}).$$

Then we can consider the following diagram of functors

$$\begin{array}{ccc} \text{Mod } \Lambda & \xrightarrow{F} & \text{Gr}(\Gamma) \\ D\text{tr}_\Lambda^n \downarrow & & \downarrow D \\ \text{Mod } \Lambda & \xrightarrow{F'} & \text{Gr}(\Gamma^{\text{op}}) \end{array}$$

The following result shows the above mentioned intimate connection with Frobenius algebras.

**Proposition 2.2.** *Let  $\Lambda$  be a generalized Artin-Schelter regular ring of dimension  $n$  with a duality  $D$ . Let  $T$  be the direct sum a finite subset of all simple  $\Lambda$ -modules in  $\mathcal{S}^n$  such that  $D\text{tr}_\Lambda^n$  maps  $\text{add } T$  to  $\text{add } T$ . Denote by  $\mathcal{F}(\text{add } T)$  all modules of finite length with composition factors only from  $\text{add } T$ .*

(a) *There is a commutative diagram of functors*

$$\begin{array}{ccc} \mathcal{F}(\text{add } T) & \xrightarrow{F=\text{Ext}_\Lambda^*(-, T)} & \text{gr}(\Gamma) \\ D\text{tr}_\Lambda^n \downarrow & & \downarrow D \\ \mathcal{F}(\text{add } T) & \xrightarrow{F'=\text{Ext}_\Lambda^*(T, -)} & \text{gr}(\Gamma^{\text{op}}) \end{array}$$

- (b) *Let  $T$  be such that  $D\text{tr}_\Lambda^n$  induces a permutation of the simples in  $\text{add } T$ . Then the Yoneda algebra  $\Gamma = \coprod_{i \geq 0} \text{Ext}_\Lambda^i(T, T)$  is Frobenius.*  
(c) *Let  $T$  be such that  $D\text{tr}_\Lambda^n$  induces a permutation of the simples in  $\text{add } T$ . Then there is a commutative diagram of functors*

$$\begin{array}{ccc} \mathcal{F}(\text{add } T) & \xrightarrow{F} & \text{gr}(\Gamma) \\ D\text{tr}_\Lambda^n \downarrow & & \downarrow \text{Hom}_\Gamma(-, \Gamma)D \\ \mathcal{F}(\text{add } T) & \xrightarrow{F} & \text{gr}(\Gamma) \end{array}$$

*Proof.* (a) We have that  $\Gamma_0 = \text{End}_\Lambda(T)$  is Noetherian and  $\Gamma_i$  is a finitely generated  $\Gamma_0$ -module for all  $i$ , so that  $\Gamma$  is a Noetherian ring (finitely generated  $\Gamma_0$ -module). Since the functor  $F$  takes semisimple modules in  $\mathcal{F}(\text{add } T)$  to projective modules (finitely generated modules),  $F$  is a half exact functor and  $\Gamma$  is Noetherian,  $F|_{\mathcal{F}(\text{add } T)}$  has its image in  $\text{gr}(\Gamma)$ . Similar arguments are used for  $F'$ . Hence the functors in the diagram end up in the categories indicated.

By Proposition 2.1 the image of the functors  $DF$  and  $F'D\text{tr}_\Lambda^n$  on a module  $M$  in  $\mathcal{F}(\text{add } T)$  are isomorphic as abelian groups. So we need to show that they in fact are isomorphic as  $\Gamma^{\text{op}}$ -modules. Recall that the isomorphism from  $DF(M)$  to  $(F'D\text{tr}_\Lambda^n)(M)$  as abelian groups is induced by the isomorphisms

$$\varphi_i: D(\text{Ext}_\Lambda^{n-i}(M, L)) \rightarrow \text{Ext}_\Lambda^i(L, D\text{tr}_\Lambda^n(M))$$

for all  $i \geq 0$ . For any element  $\theta$  in  $\text{Ext}_\Lambda^j(T, T)$  we want to prove that  $\varphi_{i+j}(f)\theta = \varphi_i(f\theta)$  for all  $f$  in  $D\text{Ext}_\Lambda^{i+j}(M, T)$ .

Represent  $\theta$  in  $\text{Ext}_\Lambda^j(T, T)$  as a homomorphism  $\theta: \Omega_\Lambda^j(T) \rightarrow T$ . This gives rise to the following diagram

$$\begin{array}{ccccc}
 D \text{Ext}_\Lambda^{i+j}(M, T) & \xrightarrow{D \text{Ext}_\Lambda^{i+j}(M, \theta)} & D \text{Ext}_\Lambda^{i+j}(M, \Omega_\Lambda^j(T)) & \longrightarrow & D \text{Ext}_\Lambda^i(M, T) \\
 \downarrow \varphi_{i+j} & & \downarrow \varphi_{i+j} & & \downarrow \varphi_i \\
 \text{Ext}_\Lambda^{n-i-j}(T, D\text{tr}_\Lambda^n(M)) & \xrightarrow{\text{Ext}_\Lambda^{n-i-j}(\theta, D\text{tr}_\Lambda^n(M))} & \text{Ext}_\Lambda^{n-i-j}(\Omega_\Lambda^j(T), D\text{tr}_\Lambda^n(M)) & \longrightarrow & \text{Ext}_\Lambda^{n-i}(T, D\text{tr}_\Lambda^n(M))
 \end{array}$$

The first square commutes, since  $\varphi_l$  is a natural transformation for all  $l$ . The horizontal morphisms in the second square are compositions of connecting homomorphisms or dual thereof. The isomorphisms  $\varphi_l$  are all induced by the isomorphism for a finitely generated projective module  $P$  and any module  $X$

$$\psi: D \text{Hom}_\Lambda(P, X) \rightarrow \text{Hom}_\Lambda(X, D(P^*))$$

given by  $\psi(g)(x)(p^*) = g(p^*(-)x)$  for  $x$  in  $X$  and  $p^*$  in  $P^*$ , where  $p^*(-)x$  is in  $\text{Hom}_\Lambda(P, X)$ . This is a natural isomorphism in both variables. It follows from this that the isomorphisms  $\varphi_l$  all commute with connecting morphisms, that is, if  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$  is exact, then the diagram

$$\begin{array}{ccc}
 D \text{Ext}_\Lambda^{i+1}(M, X) & \xrightarrow{D(\partial)} & D \text{Ext}_\Lambda^i(M, Z) \\
 \downarrow \varphi_{i+1} & & \downarrow \varphi_i \\
 \text{Ext}_\Lambda^{n-i-1}(X, D\text{tr}_\Lambda^n(M)) & \xrightarrow{\partial'} & \text{Ext}_\Lambda^{n-i}(Z, D\text{tr}_\Lambda^n(M))
 \end{array}$$

is commutative.

Then going back to our first diagram, which by the above is commutative, the upper diagonal path is computing  $\varphi_i(f\theta)$  while the lower diagonal path is computing  $\varphi_{i+j}(f)\theta$  both for any  $f$  in  $D \text{Ext}_\Lambda^{i+j}(M, T)$ . Hence we conclude that they are equal and  $DF(M)$  and  $F'(D\text{tr}_\Lambda^n(M))$  are isomorphic as  $\Gamma^{\text{op}}$ -modules for all  $M$  in  $\mathcal{F}(\text{add } T)$ . Since the isomorphism between  $DF$  and  $F'D\text{tr}_\Lambda^n$  is induced by the natural isomorphisms  $\{\varphi_i\}_{i=0}^n$ , we infer that  $DF$  and  $F'D\text{tr}_\Lambda^n$  are isomorphic as functors.

(b) The functor  $F: \mathcal{F}(\text{add } T) \rightarrow \text{gr}(\Gamma)$  sends simple modules to projective modules, and the functor  $D\text{tr}_\Lambda^n: \mathcal{F}(\text{add } T) \rightarrow \mathcal{F}(\text{add } T)$  sends simple modules to simple modules. From the commutative diagram in (a) we infer that the projective module  $\text{Ext}_\Lambda^*(T, D\text{tr}_\Lambda^n(S))$  is isomorphic to the injective module  $D(\text{Hom}_\Gamma(\text{Ext}_\Lambda^*(S, T), \Gamma))$ . Since  $D\text{tr}_\Lambda^n$  is a permutation of all simple modules in  $\text{add } T$ , it follows that the projective and the injective modules coincide and  $\Gamma$  is Frobenius.

(c) Using (a) it is enough to show that  $\text{Hom}_\Gamma(-, \Gamma)F'$  is isomorphic to  $F$ .

We want to consider  $F'$  as a functor  $F': E(\mathcal{F}(\text{add } T)) \rightarrow \text{gr}(\Gamma)$ . Here  $E(\mathcal{F}(\text{add } T))$  is the Ext-category of the full subcategory  $\mathcal{F}(\text{add } T)$  of  $\text{mod } \Lambda$ , that is,  $E(\mathcal{F}(\text{add } T))$  has  $\mathcal{F}(\text{add } T)$  as objects and morphisms are given by

$$\text{Hom}_{E(\mathcal{F}(\text{add } T))}(M, N) = \coprod_{i \geq 0} \text{Ext}_\Lambda^i(M, N)$$

for  $M$  and  $N$  in  $\mathcal{F}(\text{add } T)$ . On objects  $F'$  is given as usual, while on morphisms we have the following. For  $M$  and  $N$  in  $E(\mathcal{F}(\text{add } T))$ , we let for  $\theta$  in  $\text{Ext}_\Lambda^i(M, N)$

$$F'(\theta): F'(M) = \text{Ext}_\Lambda^*(T, M) \rightarrow \text{Ext}_\Lambda^*(T, N) = F'(N)$$

be given by the Yoneda multiplication by  $\theta$ . Hence in this way  $F'$  induces a morphism from  $\text{Ext}_\Lambda^*(M, N)$  to  $\text{Hom}_\Gamma(F'(M), F'(N))$ . In particular for  $N = T$  we have that  $F'$  induces a morphism from  $F(M) = \text{Ext}_\Lambda^*(M, T)$  to  $\text{Hom}_\Gamma(F'(M), F'(T)) = \text{Hom}_\Gamma(F'(M), \Gamma)$ .

The above morphism induced by  $F'$  is clearly an isomorphism for  $T$ , and therefore we have an isomorphism for all simple modules in  $\mathcal{S}^n$ . Suppose we have an isomorphism for all modules of length  $l$  in  $\mathcal{F}(\text{add } T)$ . Let  $L$  be a module with length  $l+1$  in  $\mathcal{F}(\text{add } T)$ . Then there is an exact sequence  $\eta: 0 \rightarrow L' \rightarrow L \rightarrow S \rightarrow 0$  with  $S$  a simple module in  $\mathcal{S}^n$  and  $L'$  of length  $l$  in  $\mathcal{H}_\Lambda^n$ . The long exact sequence induced by  $\eta$  applying  $\text{Hom}_\Lambda(T, -)$  and the fact that  $\Gamma$  is selfinjective by (b) induce the following commutative diagram

$$\begin{array}{ccccccccc} (F(L')[-1]) & \longrightarrow & F(S) & \longrightarrow & F(L) & \longrightarrow & F(L') & \longrightarrow & F(S)[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ F'(L')[1]^* & \longrightarrow & (F'(S))^* & \longrightarrow & (F'(L))^* & \longrightarrow & (F'(L'))^* & \longrightarrow & (F'(S)[-1])^* \end{array}$$

Since by induction the two leftmost and the two rightmost vertical maps are isomorphisms, we have by the Five Lemma that  $F(L) \simeq \text{Hom}_\Gamma(\text{Ext}_\Lambda^*(T, L), \Gamma)$ . The claim follows from this.  $\square$

For connected Koszul algebras  $\Lambda$  of finite global dimension P. Smith in [24] has shown that the Koszul dual of  $\Lambda$  is Frobenius if and only if  $\Lambda$  is Gorenstein. This result is generalized to the non-connected situation by Martínéz-Villa in [19]. Lupatieri-Wu-Zhang have shown in [22] that for a connected graded algebra  $\Lambda$  with Ext-algebra  $E(\Lambda)$ , then  $\Lambda$  is Artin-Schelter regular if and only if  $E(\Lambda)$  is Frobenius. The above result shows one direction of this result for our more general class of Artin-Schelter regular algebras. For illustration let us return to Example 1.1.

**Example 2.3.** Let  $\Sigma = K[x]/(x^3)$  as in Example 1.1, and let  $\Lambda$  be the Auslander algebra  $\text{End}_\Sigma(M)^{\text{op}}$  of  $\Sigma$  where  $M = U \amalg L \amalg \Sigma$ . Applying the functor

$$\text{Hom}_\Lambda(-, \Lambda) = \text{Hom}_\Lambda(-, \text{Hom}_\Sigma(M, M)^{\text{op}})$$

to the projective resolutions of  $S_U$  and  $S_L$  induces the following exact sequences

$$0 \rightarrow {}_\Sigma(U, M) \rightarrow {}_\Sigma(L, M) \rightarrow {}_\Sigma(U, M) \rightarrow \text{tr}_\Lambda^2(S_U) \rightarrow 0$$

and

$$0 \rightarrow {}_\Sigma(L, M) \rightarrow {}_\Sigma(\Sigma \amalg U, M) \rightarrow {}_\Sigma(L, M) \rightarrow \text{tr}_\Lambda^2(S_L) \rightarrow 0,$$

where  $\text{tr}_\Lambda^2(S_U) \simeq S_U^{\text{op}}$  and  $\text{tr}_\Lambda^2(S_L) \simeq S_L^{\text{op}}$ . Since  $D(S^{\text{op}}) \simeq S$  for all simple module  $S$  (and  $S^{\text{op}}$  over  $\Lambda^{\text{op}}$ ), it follows that  $D\text{tr}_\Lambda^2(S) \simeq S$  for  $S = S_U, S_L$ . Hence we see that the condition in Proposition 2.2 (b) is satisfied, and therefore  $\Gamma = \text{Ext}_\Lambda^*(T, T)$  is Frobenius for  $T = S_U \amalg S_L$ , as pointed out already in Example 1.1.

For an Auslander algebra  $\Lambda$  of a finite dimensional algebra  $\Sigma$  of finite representation type, it is straightforward to see that  $D\text{tr}_\Lambda^2(S_C)$  for some indecomposable  $\Sigma$ -module  $C$  is given by  $S_{\tau C}$ , where  $\tau$  is the Auslander-Reiten translate. Hence to find a finite set of simple modules satisfying the assumptions in Proposition 2.2 (b), it is enough to choose simple modules corresponding to indecomposable  $\Sigma$ -modules permuted by  $\tau$ .

Now we make a further remark concerning Proposition 2.2. Recall that the functor  $\text{Hom}_\Gamma(-, \Gamma)D$  is the Nakayama functor for the Frobenius algebra  $\Gamma$ . Hence,

Proposition 2.2 (c) in other words says that  $D\text{tr}_\Lambda^n$  essentially behaves like the Nakayama functor via the functor  $F = \text{Ext}_\Lambda^*(-, T)$ .

For hereditary categories discussed in [23], the existence of a Serre functor is connected with the existence of almost split sequences. In Proposition 2.1 we saw that  $D\text{tr}_\Lambda^n$  plays the role of a Serre functor for generalized Artin-Schelter regular algebras. For our  $n$ -dimensional Artin-Schelter regular algebras, it gives rise to  $n$ -fold almost split extensions, as we explain next. To this end recall that a non-zero  $n$ -fold extension  $\eta$  in  $\text{Ext}_\Lambda^n(C, A)$  is called an  *$n$ -fold almost split extension* if for any non-splittable epimorphism  $f: X \rightarrow C$  and for any non-splittable monomorphism  $g: A \rightarrow Y$  the pullback  $\eta \cdot f$  and the pushout  $g \cdot \eta$  are both zero in  $\text{Ext}_\Lambda^n(X, A)$  and  $\text{Ext}_\Lambda^n(C, Y)$ , respectively. With these remarks we give a connection between the Serre functor  $D\text{tr}_\Lambda^n$  and existence of  $n$ -fold almost split extensions for the category  $\mathcal{H}_\Lambda^n$  for generalized Artin-Schelter regular algebras.

**Proposition 2.4.** *Let  $\Lambda$  be a generalized Artin-Schelter regular ring of dimension  $n$  with duality  $D$ . Assume that  $\mathcal{H}_\Lambda^n$  is a Krull-Schmidt category.*

- (a) *If  $n = 1$ , then the category  $\mathcal{H}_\Lambda^1$  has almost split sequences.*
- (b) *If  $n > 1$ , then the category  $\mathcal{H}_\Lambda^n$  has  $n$ -fold almost split extensions.*

*Proof.* From Proposition 2.1 we have that

$$D(\text{End}_\Lambda(M)) \simeq \text{Ext}_\Lambda^n(M, D\text{tr}_\Lambda^n(M))$$

for any module  $M$  in  $\mathcal{H}_\Lambda^n$ . If  $M$  is indecomposable, then  $\text{End}_\Lambda(M)$  is a local ring. Then using standard arguments as in [5] both claims in (a) and (b) follows.  $\square$

### 3. THE POLYNOMIAL RING

In this section we consider the polynomial ring  $\Lambda = K[x_1, \dots, x_n]$  in  $n$  indeterminants  $x_1, \dots, x_n$  over an algebraically closed field  $K$ . The aim is to illustrate some of the aspects discussed in the previous section on this concrete example.

First we show that  $\Lambda$  is a generalized Artin-Schelter regular ring of dimension  $n$  as a ungraded ring. The simple modules over  $\Lambda$  are well-known. Since  $K$  is algebraically closed, all maximal ideals are of the form

$$\mathfrak{m}_{\bar{a}} = (x_1 - a_1, x_2 - a_2, \dots, x_n - a_n)$$

for some  $n$ -tuple  $\bar{a} = (a_1, a_2, \dots, a_n)$  in  $K^n$ . Hence all simple  $\Lambda$ -modules are given as  $S_{\bar{a}} = \Lambda/\mathfrak{m}_{\bar{a}}$ .

For any  $n$ -tuple  $\bar{a} = (a_1, a_2, \dots, a_n)$  in  $K^n$  there is an automorphism  $\sigma_{\bar{a}}: \Lambda \rightarrow \Lambda$  given by  $\sigma_{\bar{a}}(x_i) = x_i - a_i$  for all  $i = 1, 2, \dots, n$ .

For a  $\Lambda$ -module  $M$  and for an automorphism  $\sigma: \Lambda \rightarrow \Lambda$  denote by  $M^\sigma$  the  $\Lambda$ -module with underlying vector space  $M$  and where the action of an element  $\lambda$  in  $\Lambda$  is given by  $\lambda \cdot m = \sigma^{-1}(\lambda)m$ . This extends to a functor  $\sigma: \text{Mod } \Lambda \rightarrow \text{Mod } \Lambda$ , which is an exact functor. Since the functor  $\text{Hom}_\Lambda(X^\sigma, -) \simeq \text{Hom}_\Lambda(X, -)\sigma^{-1}(-)$  for an automorphism of  $\Lambda$ , the functor  $\sigma$  preserves projective modules and in particular free modules.

With the above convention it follows that  $S_{\bar{a}} \simeq K^{\sigma_{\bar{a}}}$  for all  $n$ -tuples  $\bar{a}$  in  $K^n$ . In particular we have that  $K \simeq S_0$  with  $0 = (0, 0, \dots, 0)$ .

Let

$$\mathbb{P}: 0 \rightarrow P_n \rightarrow P_{n-1} \rightarrow \dots \rightarrow P_1 \rightarrow P_0 \rightarrow S_0 \rightarrow 0$$

be a minimal graded projective resolution (the Koszul complex) of  $S_0 = K$  over  $\Lambda$ . This is a projective resolution of  $S_0$  also when we forget the grading, and we

furthermore have that  $\text{Ext}_\Lambda^i(S_0, \Lambda) = (0)$  for  $i \neq n$  and  $\text{Ext}_\Lambda^n(S_0, \Lambda) \simeq S_0$ . Let  $\bar{a} = (a_1, a_2, \dots, a_n)$  be in  $K^n$ . Then  $\mathbb{P}^{\sigma\bar{a}}$  is a projective resolution of  $S_{\bar{a}}$ . Since  $\Lambda^{\sigma\bar{a}} \simeq \Lambda$  for all  $n$ -tuples  $\bar{a}$  and  $\text{Hom}_\Lambda(X^\sigma, \Lambda) \simeq \text{Hom}_\Lambda(X, \Lambda^{\sigma^{-1}})$ , we infer that  $\text{Ext}_\Lambda^i(S_{\bar{a}}, \Lambda) = (0)$  for all  $i \neq n$  and  $\text{tr}_\Lambda^n(S_{\bar{a}}) = \text{Ext}_\Lambda^n(S_{\bar{a}}, \Lambda) \simeq S_{\bar{a}}$ . Consider as before the subcategory

$$\mathcal{H}_\Lambda^n = \{M \in \text{mod } \Lambda \mid \text{pd}_\Lambda M = n, \text{Ext}_\Lambda^i(M, \Lambda) = (0) \text{ for all } i \neq 0\}.$$

By the above observations all simple  $\Lambda$ -modules are in  $\mathcal{H}_\Lambda^n$  and  $\text{tr}_\Lambda^n$  is the identity ( $\Lambda = \Lambda^{\text{op}}$ ), hence  $\Lambda$  is a generalized Artin-Schelter regular  $K$ -algebra as a ungraded ring. As a consequence of the above and Proposition 1.8 we have the following.

**Theorem 3.1.** *Let  $\Lambda = K[x_1, x_2, \dots, x_n]$  be the polynomial ring in  $n$  indeterminants  $x_1, x_2, \dots, x_n$  over an algebraically closed field  $K$ . Then  $\mathcal{H}_\Lambda^n$  is exactly the full subcategory consisting of all  $\Lambda$ -modules of finite length in  $\text{Mod } \Lambda$ .*

Now let us discuss Proposition 2.2 (b) for this example. In particular it says that the Yoneda algebra  $\coprod_{i \geq 0} \text{Ext}_\Lambda^i(K, K)$  is Frobenius. This is well-known and it is in fact isomorphic to the exterior algebra on  $K^n$ . However we can consider  $\Lambda$  as an ungraded  $K$ -algebra, and we have seen that  $\Lambda$  is a generalized Artin-Schelter regular  $K$ -algebra. Consider two simple  $\Lambda$ -modules  $S_{\bar{a}}$  and  $S_{\bar{b}}$ . Then each  $\text{Ext}_\Lambda^i(S_{\bar{a}}, S_{\bar{b}})$  is a module over  $\Lambda$ , where the action of  $\Lambda$  can be taken through  $S_{\bar{a}}$  or  $S_{\bar{b}}$ . Hence  $\text{Ext}_\Lambda^i(S_{\bar{a}}, S_{\bar{b}})$  is annihilated by  $\mathfrak{m}_{\bar{a}} + \mathfrak{m}_{\bar{b}} = \Lambda$  if  $\bar{a} \neq \bar{b}$ . Therefore the Ext-algebra of any finite number of simple modules over  $\Lambda$  is just a direct sum of the Ext-algebras of the simple  $\Lambda$ -modules involved. Since  $S_{\bar{a}} \simeq K^{\sigma\bar{a}}$ , it follows that each of these Ext-algebras are isomorphic to the exterior algebra. And hence Frobenius as predicted by Proposition 2.2 (b).

Finally we point out that  $\mathcal{H}_\Lambda^n$  as  $n$ -fold almost split sequences. Since  $\text{Ext}_\Lambda^i(S_{\bar{a}}, S_{\bar{b}}) = (0)$  for all  $i \geq 0$  whenever  $\bar{a} \neq \bar{b}$ , any object of finite length in  $\mathcal{H}_\Lambda^n$  is a module over  $\Lambda/\mathfrak{m}_{\bar{a}}^n$  for some  $\bar{a}$  and some positive integer  $n$ . Since this is a finite dimensional algebra, it follows that  $\mathcal{H}_\Lambda^n$  is a Krull-Schmidt category. As a corollary of Proposition 2.4 we have the following.

**Corollary 3.2.** *Let  $\Lambda = K[x_1, x_2, \dots, x_n]$  be the polynomial ring in  $n$  indeterminants over an algebraically closed field  $K$ . Then the category of  $\Lambda$ -modules of finite length  $\mathcal{H}_\Lambda^n$  has  $n$ -fold almost split sequences.*

#### 4. NON-NOETHERIAN GENERALIZED ARTIN-SCHELTER REGULAR ALGEBRAS

It is unknown whether or not (to our knowledge) that all connected Artin-Schelter regular algebras are Noetherian. In our definition a generalized Artin-Schelter regular algebra we have not put any requirement on the Gelfand-Kirillov dimension of the algebra. So this is maybe why we in this section can exhibit examples of generalized Artin-Schelter regular algebras which are not Noetherian. However, they are Noetherian if and only if the Gelfand-Kirillov dimension is finite.

To give the class of generalized Artin-Schelter regular algebras we construct in this section, we recall for the convenience of the reader the results we used from [7, 8, 16, 17, 18]. The first result we recall characterizes selfinjective Koszul algebras in terms of the Koszul dual.

**Theorem 4.1** ([16]). *Let  $\Lambda$  be a Koszul algebra with Yoneda algebra  $\Gamma$ . Then  $\Lambda$  is selfinjective if and only if there exists some positive integer  $n$  such that all graded simple  $\Gamma$ -modules have the same projective dimension  $n$  and  $\Gamma$  satisfies the  $n$ -simple condition.*

It is easy to find selfinjective Koszul algebras, as the following theorem shows.

**Theorem 4.2** ([17]). *Let  $\Lambda$  be an indecomposable selfinjective finite dimensional  $K$ -algebra with radical  $\mathfrak{r}$  such that  $\mathfrak{r}^2 \neq 0$  and  $\mathfrak{r}^3 = 0$ . Then  $\Lambda$  is Koszul if and only if it is of infinite representation type. In the Koszul case the Yoneda algebra  $\Gamma$  has global dimension 2.*

**Corollary 4.3.** *Let  $Q$  be a connected bipartite graph, and let  $K$  be a field. Then the trivial extension  $\Lambda = KQ \ltimes D(KQ)$  is Koszul if and only if  $Q$  is non-Dynkin. If  $Q$  is non-Dynkin, then the Yoneda algebra  $\Gamma$  of  $\Lambda$  is the preprojective algebra.*

We can characterize when an Artin-Schelter regular Koszul algebra of global dimension 2 is Noetherian.

**Theorem 4.4** ([7, 8]). *Let  $\Gamma$  be an indecomposable Koszul algebra, where all the graded simple  $\Gamma$ -modules have projective dimension 2 and satisfying the 2-simple condition. Suppose  $\Lambda$  is the Yoneda algebra of  $\Gamma$ . If  $\Lambda$  is a tame algebra, then  $\Gamma$  is Noetherian of Gelfand-Kirillov dimension two. If  $\Lambda$  is wild, then  $\Gamma$  is non-Noetherian of infinite Gelfand-Kirillov dimension.*

It is easy to built new Artin-Schelter regular algebras from given ones, this is the situation considered in next theorem.

**Theorem 4.5** ([18]). *Let  $\Lambda$  be a Koszul  $K$ -algebra with Yoneda algebra  $\Gamma$ , and let  $G$  be a finite group of automorphisms of  $\Lambda$ , such that characteristic of  $K$  does not divide the order of  $G$ . Then the following statements are true.*

- (a) *There is a natural action of  $G$  on  $\Gamma$ .*
- (b) *The skew group algebra  $\Lambda * G$  is Koszul with Yoneda algebra  $\Gamma * G$ .*
- (c) *If  $\Lambda$  is selfinjective, then  $\Lambda * G$  is selfinjective, in particular all the graded simple  $\Gamma * G$ -modules have projective dimension  $n$  and  $\Gamma * G$  satisfies the  $n$ -simple condition, where  $n$  is the Loewy length of  $\Lambda * G$ .*

Recall that when  $\Lambda$  is a graded algebra and  $G$  is a group of automorphisms acting on  $\Lambda$ , then  $\Lambda * G$  is a graded algebra again with degree  $i$  of  $\Lambda * G$  being given by  $\Lambda_i * G$ .

**Corollary 4.6.** *Let  $\Gamma = K[x_1, x_2, \dots, x_n]$  be the polynomial algebra, and let  $G$  be a finite group of automorphisms of  $\Gamma$  such that characteristic of  $K$  does not divide the order of  $G$ . Then the skew group algebra  $K[x_1, x_2, \dots, x_n] * G$  is Artin-Schelter regular.*

*Proof.* Let  $\Lambda$  be the exterior algebra of a  $K$ -vector space of dimension  $n$ . Then it is known  $\Lambda$  is selfinjective Koszul with Yoneda algebra  $\Gamma$ . Then apply Theorem 4.5, and the claim follows.  $\square$

Now we are ready to give the construction of a class of generalized Artin-Schelter

regular algebras in the sense of Section 1. Let  $Q$  be the quiver  $1 \begin{matrix} \xrightarrow{\alpha_1} \\ \vdots \alpha_2 \\ \xrightarrow{\alpha_n} \end{matrix} 2$  for

$n \geq 2$ , where we denote the trivial paths corresponding to the vertices by  $e_1$  and  $e_2$ . Consider the trivial extension  $\Lambda = KQ \triangleright D(KQ)$ . Then  $\Lambda$  is isomorphic to the algebra  $K\widehat{Q}/I$ , where  $\widehat{Q}$  is the quiver described as follows. The quiver  $\widehat{Q}$  has vertices  $\widehat{Q}_0 = Q_0$  and arrows  $\widehat{Q}_1 = Q_1 \cup Q_1^{\text{op}}$ , that is, in addition to the old arrows in  $Q$ , we introduce an arrow  $\widehat{\alpha}$  in the opposite direction for any arrow  $\alpha$  in  $Q$ . The ideal  $I$  is generated by the following relations  $\{\alpha\widehat{\alpha} - \beta\widehat{\beta}, \widehat{\alpha}\alpha - \widehat{\beta}\beta, \widehat{\alpha}\beta, \alpha\widehat{\beta}\}_{\alpha \neq \beta \in Q_1}$ .

Let the cyclic group  $G = \langle g \rangle$  of order 2 act as automorphisms of  $\Lambda$  by

$$\begin{aligned} g(e_1) &= e_2, \\ g(e_2) &= e_1, \\ g(\alpha) &= \widehat{\alpha}, \\ g(\widehat{\alpha}) &= \alpha. \end{aligned}$$

Assume the characteristic of  $K$  different from 2. Then  $\Lambda * G$  is selfinjective Koszul, and  $(\Lambda * G)_0$  has  $\{\overline{e}_i = e_i \otimes 1\}_{i=1}^2 \cup \{\overline{e}_i g = e_i \otimes g\}_{i=1}^2$  as a basis over  $K$ . Define the map  $\varphi: (\Lambda * G)_0 \rightarrow M_2(K)$  by letting

$$\begin{aligned} \varphi(\overline{e}_1) &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \\ \varphi(\overline{e}_2) &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \\ \varphi(\overline{e}_1 g) &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\ \varphi(\overline{e}_2 g) &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

One easily checks that  $\varphi$  is an isomorphism and therefore  $(\Lambda * G)_0 \simeq M_2(K)$ .

The degree one part  $(\Lambda * G)_1$  is generated by  $\langle \alpha \otimes 1, \alpha \otimes g, \widehat{\alpha} \otimes 1, \widehat{\alpha} \otimes g \rangle_{\alpha \in Q_1}$ , and  $(\Lambda * G)_2$  is generated by  $\langle \alpha \widehat{\alpha} \otimes 1, \alpha \widehat{\alpha} \otimes g, \widehat{\alpha} \alpha \otimes 1, \widehat{\alpha} \alpha \otimes g \rangle_{\alpha \in Q_1}$ . It follows that  $\Lambda * G$  is Morita equivalent to  $e_1(\Lambda * G)e_1$ . The algebra  $e_1(\Lambda * G)e_1$  is a basic connected selfinjective Koszul algebra, where

$$\begin{aligned} (e_1(\Lambda * G)e_1)_0 &= K, \\ (e_1(\Lambda * G)e_1)_1 &= \langle \widehat{\alpha} \otimes g \rangle_{\alpha \in Q}, \\ (e_1(\Lambda * G)e_1)_2 &= \langle \widehat{\alpha} \alpha \otimes 1 \rangle_{\alpha \in Q}, \end{aligned}$$

since  $e_1 \widehat{\alpha} \otimes g e_1 = e_1 \widehat{\alpha} e_2 \otimes g$  and  $(\widehat{\alpha} \otimes g)(\widehat{\alpha} \otimes g) = \widehat{\alpha} \alpha \otimes g^2 = \widehat{\alpha} \alpha \otimes 1$ . It follows from this that  $e_1(\Lambda * G)e_1$  is isomorphic to

$$\begin{aligned} A &= K\langle x_1, x_2, \dots, x_n \rangle / \langle \{x_i x_j\}_{i \neq j}, \{x_i^2 - x_2^2\}_{i \neq 2} \rangle \\ &= K[x_1, x_2, \dots, x_n] / (\{x_i x_j\}_{i \neq j}, \{x_i^2 - x_2^2\}_{i \neq 2}). \end{aligned}$$

The algebra  $A$  is selfinjective Koszul with Yoneda algebra

$$B = A^! = K\langle x_1, x_2, \dots, x_n \rangle / \langle \sum_{i=1}^n x_i^2 \rangle.$$

Since  $B$  is Morita equivalent to  $\Gamma * G$ , with  $\Gamma$  the preprojective algebra of the quiver

$$Q: 1 \xrightarrow[\alpha_n]{\begin{smallmatrix} \alpha_1 \\ \vdots \\ \alpha_2 \end{smallmatrix}} 2, \text{ then it is non-Noetherian and of Gelfand-Kirillov dimension infinite}$$

for  $n \geq 3$  by Theorem 4.5 and 4.4. We obtain the following theorem.

**Theorem 4.7.** *The algebras  $\Gamma_n = K\langle x_1, x_2, \dots, x_n \rangle / \langle \sum_{i=1}^n x_i^2 \rangle$  are Koszul for all  $n \geq 1$ , and all graded simple  $\Gamma_n$ -modules have projective dimension  $n$  and  $\Gamma_n$  satisfies the  $n$ -simple condition. For  $n = 2$  they are Noetherian of Gelfand-Kirillov*



dimension 2, and for  $n \geq 3$  they are non-Noetherian and of infinite Gelfand-Kirillov dimension.

## 5. GENERALIZED ARTIN-SCHELTER REGULAR CATEGORIES

The aim of this section is to generalize the notion of generalized Artin-Schelter regular algebras introduced in Section 1 to generalized Artin-Schelter regular categories. Furthermore, we show that the same results hold true in this situation. The need for such a generalization was already indicated in the introduction and Section 1, as our main application is the associated graded Auslander category of a component in the Auslander-Reiten quiver of a finite dimensional algebra  $\Lambda$  (See the introduction to Section 7 for further details). In this category there are as many simple objects as there are indecomposable finitely generated modules over  $\Lambda$ . Hence the need to extend to categories is apparent.

Now we proceed with describing the categorical version of generalized Artin-Schelter regular algebra, where we refer the reader to [20] for the details concerning the notion discussed below. Let  $\mathcal{C}$  be an additive  $K$ -category over a field  $K$ . Consider the category  $\text{Mod}(\mathcal{C})$  of all additive functors  $\mathcal{C}^{\text{op}} \rightarrow \text{Mod } K$ , where the finitely generated projective objects are given by  $\text{Hom}_{\mathcal{C}}(-, C)$  for an object  $C$  in  $\mathcal{C}$ . Define the functor  $(-)^*: \text{Mod}(\mathcal{C}) \rightarrow \text{Mod}(\mathcal{C}^{\text{op}})$  as follows. For  $F$  in  $\text{Mod}(\mathcal{C})$  let

$$F^*(X) = \text{Hom}_{\text{Mod}(\mathcal{C})}(F, \text{Hom}_{\mathcal{C}}(-, X))$$

for all objects  $X$  in  $\mathcal{C}$ , and for a morphism  $f: X \rightarrow Y$  in  $\mathcal{C}$  we have

$$F^*(f): F^*(X) = \text{Hom}_{\mathcal{C}}(F, \text{Hom}_{\mathcal{C}}(-, X)) \rightarrow \text{Hom}_{\mathcal{C}}(F, \text{Hom}_{\mathcal{C}}(-, Y)) = F^*(Y)$$

given by  $\text{Hom}_{\mathcal{C}}(F, \text{Hom}_{\mathcal{C}}(-, f))$ . For a morphism  $\eta: F \rightarrow F'$  in  $\text{Mod}(\mathcal{C})$  let

$$\eta^*(X) = \text{Hom}_{\text{Mod}(\mathcal{C})}(\eta, \text{Hom}_{\mathcal{C}}(-, X)): (F')^*(X) \rightarrow F^*(X).$$

In particular we see that  $\text{Hom}_{\mathcal{C}}(-, C)^* = \text{Hom}_{\mathcal{C}}(C, -)$  for all objects  $C$  in  $\mathcal{C}$ . Furthermore, for any  $F$  in  $\text{Mod}(\mathcal{C})$  there is an isomorphism

$$\text{Hom}_{\mathcal{C}}(-, C)^* \otimes_{\mathcal{C}} F \simeq \text{Hom}_{\text{Mod}(\mathcal{C})}(\text{Hom}_{\mathcal{C}}(-, C), F)$$

for any object  $C$  in  $\mathcal{C}$ .

Define for any integer  $i \geq 0$  a functor  $\text{tr}_{\mathcal{C}}^i: \text{Mod}(\mathcal{C}) \rightarrow \text{Mod}(\mathcal{C}^{\text{op}})$  as follows. For  $F$  in  $\text{Mod}(\mathcal{C})$  let

$$\text{tr}_{\mathcal{C}}^i(F)(X) = \text{Ext}_{\text{Mod}(\mathcal{C})}^i(F, \text{Hom}_{\mathcal{C}}(-, X))$$

for any object  $X$  in  $\mathcal{C}$ . The action on morphisms in  $\mathcal{C}^{\text{op}}$  and in  $\text{Mod}(\mathcal{C})$  are defined in the natural way.

Let  $n$  be a positive integer. Denote by  $\mathcal{H}_{\mathcal{C}}^n$  the full subcategory of  $\text{Mod}(\mathcal{C})$  consisting of all objects  $F$  with projective dimension  $n$ , a projective resolution consisting of finitely generated functors in  $\text{Mod}(\mathcal{C})$  and  $\text{tr}_{\mathcal{C}}^i(F) = 0$  for all  $i \neq n$ ,

We constantly switch between a graded and a ungraded category  $\mathcal{C}$ , and to be more compact in the presentation we use the convention that if  $\mathcal{C}$  is a graded category, then a  $\mathcal{C}$ -module is an object in  $\text{Gr}(\mathcal{C})$ , and if  $\mathcal{C}$  is a ungraded category, then a  $\mathcal{C}$ -module is an object in  $\text{Mod}(\mathcal{C})$ .

With these preliminaries we have the analogue of Proposition 1.2 from the algebra situation. The proof is literary the same and it is left to the reader.

**Proposition 5.1.** (a) *The subcategory  $\mathcal{H}_{\mathcal{C}}^n$  is closed under extensions. If the category of  $\mathcal{C}$ -modules has global dimension  $n$ , then  $\mathcal{H}_{\mathcal{C}}^n$  is closed under cokernels of monomorphisms.*

- (b) The functor  $\mathrm{tr}_{\mathcal{C}}^n: \mathrm{Mod}(\mathcal{C}) \rightarrow \mathrm{Mod}(\mathcal{C}^{\mathrm{op}})$  restricts to a functor  $\mathrm{tr}_{\mathcal{C}}^n: \mathcal{H}_{\mathcal{C}}^n \rightarrow \mathcal{H}_{\mathcal{C}^{\mathrm{op}}}^n$ , and here it is an exact duality.
- (c) For any object  $M$  in  $\mathcal{H}_{\mathcal{C}}^n$  and any  $\mathcal{C}$ -module  $L$  there are isomorphisms

$$\mathrm{Tor}_i^{\mathrm{Mod}(\mathcal{C})}(\mathrm{tr}_{\mathcal{C}}^n(M), L) \simeq \mathrm{Ext}_{\mathrm{Mod}(\mathcal{C})}^{n-i}(M, L)$$

for all  $i \geq 0$ .

Similarly as for algebras,  $\mathrm{tr}_{\mathcal{C}^{\mathrm{op}}}^n(S)$  is not necessarily a simple object again, but the following lemma give a sufficient condition to ensure this. The proof is the same as for algebras, and it is left to the reader.

**Lemma 5.2.** *Let  $\mathcal{C}$  be an additive  $K$ -category with finite global dimension  $n$ , and let  $S$  be a simple object in  $\mathcal{H}_{\mathcal{C}^{\mathrm{op}}}^n$ . Assume that  $\mathrm{tr}_{\mathcal{C}^{\mathrm{op}}}^n(S)$  contains a simple object in  $\mathcal{H}_{\mathcal{C}}^n$  as a subobject. Then  $\mathrm{tr}_{\mathcal{C}^{\mathrm{op}}}^n(S)$  is a simple object (and it is in  $\mathcal{H}_{\mathcal{C}}^n$ ).*

To proceed we need to put further conditions on the categories. In particular, information about the simple objects and projective covers. If  $\mathcal{C}$  is a positively graded Krull-Schmidt  $K$ -category with  $\mathrm{rad}(-, -) = \prod_{i \geq 1} \mathrm{Hom}_{\mathcal{C}}(-, -)_i$ , we observed in [20, Lemma 2.3] that any bounded below functor  $F$  has a projective cover  $P \rightarrow F$  with  $P/\mathrm{rad} P \simeq F/\mathrm{rad} F$ . If  $\mathcal{C}$  is a (positively graded) Krull-Schmidt  $K$ -category, we showed in [20, Lemma 2.4] that the all simple objects are exactly those of the form  $\mathrm{Hom}_{\mathcal{C}}(-, C)/\mathrm{rad} \mathrm{Hom}_{\mathcal{C}}(-, C)$  for an indecomposable object  $C$  in  $\mathcal{C}$ .

When  $\mathcal{C}$  is a Krull-Schmidt  $K$ -category, then the full subcategory  $\mathrm{mod} \mathcal{C}$  of  $\mathrm{Mod}(\mathcal{C})$  consisting of the finitely presented objects all have projective covers. In addition if  $P \rightarrow F$  is a projective cover of a finitely presented object  $F$ , then  $P/\mathrm{rad} P \simeq F/\mathrm{rad} F$  as above. Moreover we have Nakayama's Lemma in this setting.

Let  $\mathcal{S}^n$  denote the simple  $\mathcal{C}$ -modules which are in  $\mathcal{H}_{\mathcal{C}}^n$ . Then  $\mathcal{H}_{\mathcal{C}}^n$  contains all objects of finite length with composition factors only in  $\mathcal{S}^n$  by Proposition 5.1. We continue to mimic the algebra situation, and next we want to find sufficient conditions on  $\mathcal{C}$  such that  $\mathcal{H}_{\mathcal{C}}^n$  is exactly the full category of  $\mathcal{C}$ -modules with finite length and composition factors only in  $\mathcal{S}^n$ .

Define a subfunctor  $t^n: \mathrm{Mod}(\mathcal{C}) \rightarrow \mathrm{Mod}(\mathcal{C})$  of the identity functor as follows. Let  $M$  be a  $\mathcal{C}$ -module. Then let  $t^n(M) = \sum_{L \subseteq M} L$ , where the sum is taken over all subobjects  $L$  of  $M$ , where  $L$  has finite length with composition factors only in  $\mathcal{S}^n$ . We say that a  $\mathcal{C}$ -module  $M$  is  $\mathcal{H}_{\mathcal{C}}^n$ -torsion if  $M = t^n(M)$ . A  $\mathcal{C}$ -module  $M$  is  $\mathcal{H}_{\mathcal{C}}^n$ -torsion free if  $t^n(M) = (0)$ .

**Lemma 5.3.** *Let  $\mathcal{C}$  be a (positively graded) Krull-Schmidt (locally finite)  $K$ -category such that all simple  $\mathcal{C}$ -modules are finitely presented. Then  $t^n(M/t^n(M)) = (0)$  for all  $\mathcal{C}$ -modules  $M$ , or equivalently  $M/t^n(M)$  is  $\mathcal{H}_{\mathcal{C}}^n$ -torsion free.*

*Proof.* Let  $M$  be a  $\mathcal{C}$ -module and assume that  $M/t^n(M)$  has a subfunctor of finite length with composition factors only in  $\mathcal{S}^n$ . Then  $M/t^n(M)$  has a simple subobject  $S$  from  $\mathcal{S}^n$ . Let  $S'$  be the pullback of  $S$  in  $M$ , so that we have an exact sequence

$$0 \rightarrow t^n(M) \rightarrow S' \rightarrow S \rightarrow 0.$$

It induces the following commutative exact diagram.

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \text{rad}_{\mathcal{C}}(-, C) & \longrightarrow & (-, C) & \longrightarrow & S \longrightarrow 0 \\
 & & \downarrow h & & \downarrow f & & \parallel \\
 0 & \longrightarrow & t^n(M) & \longrightarrow & S' & \longrightarrow & S \longrightarrow 0
 \end{array}$$

Since  $\text{rad}_{\mathcal{C}}(-, C)$  is finitely generated,  $\text{Im } h$  is of finite length with composition factors in  $\mathcal{S}^n$ . We infer that  $\text{Im } f$  has finite length with composition factors only in  $\mathcal{S}^n$ , so that  $\text{Im } f \subseteq t^n(M)$ . This is a contradiction, so that  $t^n(M/t^n(M)) = (0)$ .  $\square$

As for modules, we need that the finitely generated  $\mathcal{H}_{\mathcal{C}}^n$ -torsion free objects have projective dimension at most one less than the global dimension, as we show next.

**Lemma 5.4.** *Let  $\mathcal{C}$  be a (positively graded) Krull-Schmidt (locally finite)  $K$ -category such that the category of  $\mathcal{C}$ -modules has finite global dimension  $n$  and that all simple  $\mathcal{C}$ -modules are finitely presented. Suppose that  $\text{Mod}(\mathcal{C})$  is a Noetherian category, or when  $\mathcal{C}$  is graded that  $\text{rad} = \text{Hom}_{\mathcal{C}}(-, -)_{\geq 1}$ . Suppose that all simple  $\mathcal{C}^{\text{op}}$ -modules of maximal projective dimension are in  $\mathcal{H}_{\mathcal{C}^{\text{op}}}^n$  and that  $\text{tr}_{\mathcal{C}^{\text{op}}}^n(S)$  contains a simple  $\mathcal{C}$ -module from  $\mathcal{H}_{\mathcal{C}}^n$  as a subobject for all simple  $\mathcal{C}^{\text{op}}$ -modules  $S$  of maximal projective dimension  $n$ . Then any finitely generated  $\mathcal{H}_{\mathcal{C}}^n$ -torsion free  $\mathcal{C}$ -module has projective dimension at most  $n - 1$ .*

*Proof.* Let  $L$  be a finitely generated torsion free  $\mathcal{C}$ -module. As for modules we obtain  $\text{Tor}_n^{\text{Mod}(\mathcal{C})}(\text{Hom}_{\mathcal{C}}(C, -)/\text{rad Hom}_{\mathcal{C}}(C, -), L) = (0)$  for all indecomposable objects  $C$  in  $\mathcal{C}$ . Hence we infer that  $\Omega_{\mathcal{C}}^n(L)/\text{rad } \Omega_{\mathcal{C}}^n(L) = (0)$ . By Nakayama's Lemma [20, Lemma 1.10] we have that  $\Omega_{\mathcal{C}}^n(L) = (0)$ , and the projective dimension of  $L$  is at most  $n - 1$ .  $\square$

The proof of the following lemma is also the same as the one for the module situation. It shows that any  $\mathcal{H}_{\mathcal{C}}^n$ -torsion  $\mathcal{C}$ -module  $M$  satisfies the same Ext-vanishing conditions as  $\mathcal{C}$ -modules in  $\mathcal{H}_{\mathcal{C}}^n$ , namely  $\text{tr}_{\mathcal{C}}^i(M) = (0)$  for all  $i \neq n$ .

**Lemma 5.5.** *Let  $\mathcal{C}$  be a (positively graded) Krull-Schmidt (locally finite)  $K$ -category such that the category of  $\mathcal{C}$ -modules has finite global dimension  $n$  and all simple  $\mathcal{C}$ -modules are finitely presented. Suppose that all simple  $\mathcal{C}$ -modules of maximal projective dimension are in  $\mathcal{H}_{\mathcal{C}}^n$ . Then any  $\mathcal{H}_{\mathcal{C}}^n$ -torsion  $\mathcal{C}$ -module  $M$  satisfies  $\text{tr}_{\mathcal{C}}^i(M) = (0)$  for all  $i \neq n$ .*

The fact that  $\mathcal{H}_{\mathcal{C}}^n$  consists exactly of the objects of finite length now follows as for modules.

**Proposition 5.6.** *Let  $\mathcal{C}$  be a (positively graded) Krull-Schmidt (locally finite)  $K$ -category such that the category of  $\mathcal{C}$ -modules has finite global dimension  $n$  and that all simple  $\mathcal{C}$ -modules are finitely presented. Suppose that  $\text{Mod}(\mathcal{C})$  is Noetherian category, or when  $\mathcal{C}$  is graded  $\text{rad} = \text{Hom}_{\mathcal{C}}(-, -)_{\geq 1}$ . Suppose that all simple  $\mathcal{C}$ -modules and  $\mathcal{C}^{\text{op}}$ -modules of maximal projective dimension are in  $\mathcal{H}_{\mathcal{C}^{\text{op}}}^n$  and  $\mathcal{H}_{\mathcal{C}}^n$ , respectively, and if  $S$  is a simple  $\mathcal{C}^{\text{op}}$ -module in  $\mathcal{H}_{\mathcal{C}^{\text{op}}}^n$ , then  $\text{tr}_{\mathcal{C}^{\text{op}}}^n(S)$  contains a simple  $\mathcal{C}$ -module from  $\mathcal{H}_{\mathcal{C}}^n$  as a subobject for all simple  $\mathcal{C}^{\text{op}}$ -modules  $S$  of maximal projective dimension  $n$ .*

*Then the category  $\mathcal{H}_{\mathcal{C}}^n$  consists exactly of the  $\mathcal{C}$ -modules of finite length with composition factors only in  $\mathcal{S}^n$ .*

This leads to the following definition of a generalized Artin-Schelter regular category.

**Definition 5.7.** Let  $\mathcal{C}$  be a (positively graded) Krull-Schmidt (locally finite)  $K$ -category such that the category of  $\mathcal{C}$ -modules has finite global dimension  $n$ . Let  $\mathcal{C}$  be a Noetherian category, or let  $\mathcal{C}$  be a positively graded such that  $\text{rad} = \text{Hom}_{\mathcal{C}}(-, -)_{\geq 1}$ .

Then  $\mathcal{C}$  is *generalized Artin-Schelter regular (of dimension  $n$ )* if  $\mathcal{H}_{\mathcal{C}}^n$  and  $\mathcal{H}_{\mathcal{C}^{\text{op}}}^n$  contain all simple  $\mathcal{C}$ -modules and  $\mathcal{C}^{\text{op}}$ -modules of maximal projective dimension  $n$ , respectively, and  $\text{tr}^n$  maps these simple  $\mathcal{C}$ -modules to objects having a simple  $\mathcal{C}$ -module from  $\mathcal{H}^n$  on the opposite side as a subobject.

The basic elementary properties of generalized Artin-Schelter regular categories are the following.

**Corollary 5.8.** *Suppose that  $\mathcal{C}$  is generalized Artin-Schelter regular  $K$ -category of dimension  $n$ . Then the following assertions hold.*

- (a) *The categories  $\mathcal{H}_{\mathcal{C}}^n$  and  $\mathcal{H}_{\mathcal{C}^{\text{op}}}^n$  consist exactly of the objects of finite length with composition factors only in simple  $\mathcal{C}$ -modules of maximal projective dimension  $n$ .*
- (b) *The functors  $\text{tr}_{\mathcal{C}}^n: \mathcal{H}_{\mathcal{C}}^n \rightarrow \mathcal{H}_{\mathcal{C}^{\text{op}}}^n$  and  $\text{tr}_{\mathcal{C}^{\text{op}}}^n: \mathcal{H}_{\mathcal{C}^{\text{op}}}^n \rightarrow \mathcal{H}_{\mathcal{C}}^n$  are inverse dualities, which preserve length, and in particular give rise to a bijection between the simple  $\mathcal{C}$ -modules and simple  $\mathcal{C}^{\text{op}}$ -modules of maximal projective dimension  $n$ .*

We end this section with an example of a generalized Artin-Schelter regular category pointed out to us by Osamu Iyama.

**Example 5.9.** Let  $\Lambda$  be a non-semisimple finite dimensional algebra over a field  $k$ . Then a finitely generated left  $\Lambda$ -module  $M$  is called an  *$n$ -cluster tilting module* ( $(n-1)$ -maximal orthogonal module, see [10, 11]) if

$$\begin{aligned} \text{add } M &= \{X \in \text{mod } \Lambda \mid \text{Ext}_{\Lambda}^i(X, M) = (0) \text{ for } 0 < i < n\} \\ &= \{X \in \text{mod } \Lambda \mid \text{Ext}_{\Lambda}^i(M, X) = (0) \text{ for } 0 < i < n\}. \end{aligned}$$

Then  $\text{add } M$  is clearly a Krull-Schmidt category. Since  $\text{Mod}(\text{add } M)$  is equivalent to  $\text{Mod } \text{End}_{\Lambda}(M)$ , the category  $\text{add } M$  is Noetherian. By [10, Proposition 3.5.1] it follows that  $\text{add } M$  is a generalized Artin-Schelter regular category of dimension  $n+1$ .

## 6. PROPERTIES OF GENERALIZED ARTIN-SCHELTER REGULAR CATEGORIES

This section is the categorical version of Section 2 discussing Serre duality formulas and the relationship with finite length Frobenius categories. We show that analogous results as for the algebra setting are true in categorical setting, in particular that the Ext-category of any set of simple  $\mathcal{C}$ -modules in  $\mathcal{H}_{\mathcal{C}}^n$  for a generalized Artin-Schelter regular  $K$ -category of dimension  $n$  with duality is a finite length Frobenius  $K$ -category. In addition, in the Koszul case we give a converse of this.

First we consider the Serre duality formulas for generalized Artin-Schelter regular categories. Let  $\mathcal{C}$  be a generalized Artin-Schelter regular  $K$ -category of dimension  $n$ . Then we have seen that the subcategories  $\mathcal{H}_{\mathcal{C}}^n$  and  $\mathcal{H}_{\mathcal{C}^{\text{op}}}^n$  consist exactly of the objects of finite length with composition factors consisting of only simple  $\mathcal{C}$ -modules of maximal projective dimension  $n$ . Moreover,  $\mathcal{H}_{\mathcal{C}}^n$  and  $\mathcal{H}_{\mathcal{C}^{\text{op}}}^n$  are abelian categories

and  $\text{tr}_{\mathcal{C}}^n: \mathcal{H}_{\mathcal{C}}^n \rightarrow \mathcal{H}_{\mathcal{C}^{\text{op}}}^n$  is an exact duality. If in addition all simple  $\mathcal{C}$ -modules  $S$  are finite dimensional over  $K$  (in particular  $S \simeq D^2(S)$ ), the duality sends a simple  $\mathcal{C}$ -module to a simple  $\mathcal{C}^{\text{op}}$ -module, but the duality  $D$  does not necessarily induce a duality from  $\mathcal{H}_{\mathcal{C}}^n$  to  $\mathcal{H}_{\mathcal{C}^{\text{op}}}^n$ . In any case we then say that  $\mathcal{C}$  is a generalized Artin-Schelter regular  $K$ -category of dimension  $n$  with duality  $D$ . As for algebras we have the following Serre duality formulas.

**Lemma 6.1.** *Let  $\mathcal{C}$  be a generalized Artin-Schelter regular  $K$ -category of dimension  $n$  with duality  $D$ .*

- (a) *The category  $\mathcal{C}^{\text{op}}$  is a generalized Artin-Schelter regular  $K$ -category of dimension  $n$ .*
- (b) *For all  $i$  there are natural isomorphisms*

$$\text{Ext}_{\text{Mod}(\mathcal{C})}^i(M, L) \simeq \text{Tor}_{n-i}^{\text{Mod}(\mathcal{C})}(\text{tr}_{\mathcal{C}}^n(M), L)$$

and

$$\varphi_i: D(\text{Ext}_{\text{Mod}(\mathcal{C})}^i(M, L)) \rightarrow \text{Ext}_{\text{Mod}(\mathcal{C})}^{n-i}(L, D\text{tr}_{\mathcal{C}}^n(M))$$

for all  $i \geq 0$ , for any  $L$  in  $\text{Mod}(\mathcal{C})$  and any object  $M$  in  $\mathcal{H}_{\mathcal{C}}^n$ .

*Proof.* (a) From the definition it is clear that we only need to show that the category of  $\mathcal{C}^{\text{op}}$ -modules has global dimension  $n$ . For the graded case this is a consequence of [20, Theorem 1.18]. For the other case it follows in a similar way.

(b) The proof is literary the same as in the algebra case.  $\square$

Let  $\mathcal{C}$  be a generalized Artin-Schelter regular  $K$ -category of dimension  $n$  with duality  $D$ . Denote by  $\mathcal{T}$  a full additive subcategory generated by some simple  $\mathcal{C}$ -modules of maximal projective dimension  $n$ , and consider the associated Ext-category  $E(\mathcal{T})$  of  $\mathcal{T}$ . Recall that  $E(\mathcal{T})$  has the same objects as  $\mathcal{T}$ , while the morphisms for  $A$  and  $B$  in  $E(\mathcal{T})$  are given by

$$\text{Hom}_{E(\mathcal{T})}(A, B) = \bigoplus_{i \geq 0} \text{Ext}_{\text{Mod}(\mathcal{C})}^i(A, B).$$

Then define the natural functors

$$F: \text{Mod}(\mathcal{C}) \rightarrow \text{Gr}(E(\mathcal{T}))$$

and

$$F': \text{Mod}(\mathcal{C}) \rightarrow \text{Gr}(E(\mathcal{T})^{\text{op}})$$

given by

$$F(G) = \text{Ext}_{\text{Mod}(\mathcal{C})}^*(G, -): E(\mathcal{T}) \rightarrow \text{Gr}(K)$$

and

$$F'(G') = \text{Ext}_{\text{Mod}(\mathcal{C})}^*(-, G'): E(\mathcal{T})^{\text{op}} \rightarrow \text{Gr}(K).$$

Then we can consider the following diagram of functors

$$\begin{array}{ccc} \text{Mod}(\mathcal{C}) & \xrightarrow{F} & \text{Gr}(E(\mathcal{T})) \\ D\text{tr}_{\mathcal{C}}^n \downarrow & & \downarrow D \\ \text{Mod}(\mathcal{C}) & \xrightarrow{F'} & \text{Gr}(E(\mathcal{T}^{\text{op}})) \end{array}$$

The following result shows that under certain stability conditions we obtain a Frobenius category from  $\mathcal{T}$ .

**Theorem 6.2.** *Let  $\mathcal{C}$  be a generalized Artin-Schelter regular  $K$ -category of dimension  $n$  with duality  $D$ . Let  $\mathcal{T}$  be a full additive subcategory generated by some simple  $\mathcal{C}$ -modules of maximal projective dimension  $n$  such that  $D\mathrm{tr}_{\mathcal{C}}^n$  maps  $\mathcal{T}$  into itself. Denote by  $\mathcal{F}(\mathcal{T})$  the full subcategory of  $\mathcal{H}_{\mathcal{C}}^n$  consisting of objects with finite length and compositions factors only in  $\mathcal{T}$ .*

(a) *There is a commutative diagram of functors*

$$\begin{array}{ccc} \mathcal{F}(\mathcal{T}) & \xrightarrow{F} & \mathrm{gr}(E(\mathcal{T})) \\ D\mathrm{tr}_{\mathcal{C}}^n \downarrow & & \downarrow D \\ \mathcal{F}(\mathcal{T}) & \xrightarrow{F'} & \mathrm{gr}(E(\mathcal{T}^{\mathrm{op}})) \end{array}$$

where  $\mathrm{gr}E(\mathcal{T})$  is a finite length category.

- (b) *Let  $\mathcal{T}$  be such that  $D\mathrm{tr}_{\mathcal{C}}^n$  induces a bijection on the simple  $\mathcal{C}$ -modules in  $\mathcal{T}$ . Then the graded  $K$ -category  $\mathrm{gr}(E(\mathcal{T}))$  is Frobenius.*
- (c) *Let  $\mathcal{T}$  be such that  $D\mathrm{tr}_{\mathcal{C}}^n$  induces a bijection on the simple  $\mathcal{C}$ -modules in  $\mathcal{T}$ . There is a commutative diagram of functors*

$$\begin{array}{ccc} \mathcal{F}(\mathcal{T}) & \xrightarrow{F} & \mathrm{gr}(E(\mathcal{T})) \\ D\mathrm{tr}_{\mathcal{C}}^n \downarrow & & \downarrow (-)^* D \\ \mathcal{F}(\mathcal{T}) & \xrightarrow{F} & \mathrm{gr}(E(\mathcal{T})) \end{array}$$

*Proof.* (a) For a projective functor  $\mathrm{Ext}_{\mathrm{Mod}(\mathcal{C})}^*(S, -)$  in  $\mathrm{gr}E(\mathcal{T})$  we have

$$\mathrm{Ext}_{\mathrm{Mod}(\mathcal{C})}^*(S, -) = \coprod_{i=0}^n \mathrm{Ext}_{\mathrm{Mod}(\mathcal{C})}^i(S, -),$$

where

$$\mathrm{Ext}_{\mathrm{Mod}(\mathcal{C})}^i(S, -) \simeq \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P_i / \mathrm{rad} P_i, -)$$

when

$$0 \rightarrow P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow S \rightarrow 0$$

is a projective resolution of  $S$ . Since  $P_i$  is finitely generated, the support of the functor  $\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P_i / \mathrm{rad} P_i, -)$  is finite. Hence the support of  $\mathrm{Ext}_{\mathrm{Mod}(\mathcal{C})}^*(S, -)$  is finite and therefore  $\mathrm{gr}E(\mathcal{T})$  is a finite length category.

Since the functor  $F$  maps semisimple  $\mathcal{C}$ -modules in  $\mathcal{F}(\mathcal{T})$  to finitely generated projective objects in  $\mathrm{gr}E(\mathcal{T})$ ,  $F$  is a half exact functor and  $\mathrm{gr}E(\mathcal{T})$  is a finite length category, therefore  $F|_{\mathcal{F}(\mathcal{T})}$  has its image in  $\mathrm{gr}E(\mathcal{T})$ . Similar arguments are used for  $F'$ . Hence the functors in the diagram end up in the categories indicated.

By Lemma 6.1 the image of the functors  $DF$  and  $F'D\mathrm{tr}_{\mathcal{C}}^n$  on a  $\mathcal{C}$ -module  $M$  in  $\mathcal{F}(\mathcal{T})$  are isomorphic as graded vector spaces. So we need to show that they in fact are isomorphic as  $E(\mathcal{T}^{\mathrm{op}})$ -modules. Recall that the isomorphism from  $DF(M)$  to  $(F'D\mathrm{tr}_{\mathcal{C}}^n)(M)$  as graded vector spaces is induced by the isomorphisms

$$\varphi_i: D(\mathrm{Ext}_{\mathrm{Mod}(\mathcal{C})}^{n-i}(M, L)) \rightarrow \mathrm{Ext}_{\mathrm{Mod}(\mathcal{C})}^i(L, D\mathrm{tr}_{\mathcal{C}}^n(M))$$

for all  $i \geq 0$ . For any element  $\theta$  in  $\mathrm{Ext}_{\mathrm{Mod}(\mathcal{C})}^j(T, T)$  we want to prove that  $\varphi_{i+j}(f)\theta = \varphi_i(f\theta)$  for all  $f$  in  $D\mathrm{Ext}_{\mathrm{Mod}(\mathcal{C})}^{i+j}(M, T)$ .

Represent  $\theta$  in  $\text{Ext}_{\text{Mod}(\mathcal{C})}^j(T, T)$  as a homomorphism  $\theta: \Omega_{\mathcal{C}}^j(T) \rightarrow T$ . This gives rise to the following diagram

$$\begin{array}{ccccc} D\text{Ext}_{\text{Mod}(\mathcal{C})}^{i+j}(M, T) & \xrightarrow{D\text{Ext}_{\text{Mod}(\mathcal{C})}^{i+j}(M, \theta)} & D\text{Ext}_{\text{Mod}(\mathcal{C})}^{i+j}(M, \Omega_{\mathcal{C}}^j(T)) & \longrightarrow & D\text{Ext}_{\text{Mod}(\mathcal{C})}^i(M, T) \\ \downarrow \varphi_{i+j} & & \downarrow \varphi_{i+j} & & \downarrow \varphi_i \\ \text{Ext}_{\text{Mod}(\mathcal{C})}^{n-i-j}(T, D\text{tr}_{\mathcal{C}}^n(M)) & \xrightarrow{\text{Ext}_{\text{Mod}(\mathcal{C})}^{n-i-j}(\theta, D\text{tr}_{\mathcal{C}}^n(M))} & \text{Ext}_{\text{Mod}(\mathcal{C})}^{n-i-j}(\Omega_{\mathcal{C}}^j(T), D\text{tr}_{\mathcal{C}}^n(M)) & \longrightarrow & \text{Ext}_{\text{Mod}(\mathcal{C})}^{n-i}(T, D\text{tr}_{\mathcal{C}}^n(M)) \end{array}$$

The first square commutes, since  $\varphi_l$  is a natural transformation for all  $l$ . The horizontal morphisms in the second square are compositions of connecting homomorphisms or dual thereof. The isomorphisms  $\varphi_l$  are all induced by the isomorphism for  $\text{Hom}_{\mathcal{C}}(-, C)$  and any  $\mathcal{C}$ -module  $G$

$$\psi: D\text{Hom}_{\text{Mod}(\mathcal{C})}(\mathcal{C}(-, C), G) \rightarrow \text{Hom}_{\text{Mod}(\mathcal{C})}(G, D(\mathcal{C}(-, C)^*))$$

given by  $\psi(g)_X(u)(s) = g(\alpha_X(u) \circ \mathcal{C}(-, s))$  for  $g$  in  $D(\text{Hom}_{\text{Mod}(\mathcal{C})}(\text{Mod}(\mathcal{C})(-, C), G))$ ,  $u$  in  $G(X)$  and  $s$  in  $\text{Hom}_{\mathcal{C}}(C, X)$  for all  $X$  in  $\mathcal{C}$ , where

$$\alpha_X: G(X) \rightarrow \text{Hom}_{\text{Mod}(\mathcal{C})}(\mathcal{C}(-, X), G)$$

is the Yoneda isomorphism. This is a natural isomorphism in both variables. It follows from this that the isomorphisms  $\varphi_l$  all commute with connecting morphisms, that is, if  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$  is exact, then the diagram

$$\begin{array}{ccc} D\text{Ext}_{\text{Mod}(\mathcal{C})}^{i+1}(M, X) & \xrightarrow{D(\partial)} & D\text{Ext}_{\text{Mod}(\mathcal{C})}^i(M, Z) \\ \downarrow \varphi_{i+1} & & \downarrow \varphi_i \\ \text{Ext}_{\text{Mod}(\mathcal{C})}^{n-i-1}(X, D\text{tr}_{\mathcal{C}}^n(M)) & \xrightarrow{\partial'} & \text{Ext}_{\text{Mod}(\mathcal{C})}^{n-i}(Z, D\text{tr}_{\mathcal{C}}^n(M)) \end{array}$$

is commutative.

Then going back to our first diagram, which by the above is commutative, the upper diagonal path is computing  $\varphi_i(f\theta)$  while the lower diagonal path is computing  $\varphi_{i+j}(f)\theta$  both for any  $f$  in  $D\text{Ext}_{\text{Mod}(\mathcal{C})}^{i+j}(M, T)$ . Hence we conclude that they are equal and  $DF(M)$  and  $F'(D\text{tr}_{\mathcal{C}}^n(M))$  are isomorphic as  $E(\mathcal{T}^{\text{op}})$ -modules for all  $M$  in  $\mathcal{F}(\mathcal{T})$ .

(b) By (a) we have that

$$D(\text{Ext}_{\text{Mod}(\mathcal{C})}^*(-, D(\text{tr}_{\mathcal{C}}^n(S)))) \simeq \text{Ext}_{\text{Mod}(\mathcal{C})}^*(S, -)$$

as objects in  $\text{gr}(E(\mathcal{T}))$ . Since  $D\text{tr}_{\mathcal{C}}^n$  is a bijection on the simple objects in  $\mathcal{T}$ , it follows that the projective and the injective objects in  $\text{gr}E(\mathcal{T})$  coincide, and  $\text{gr}E(\mathcal{T})$  is Frobenius.

(c) Recall that  $F'(G') = \text{Ext}_{\text{Mod}(\mathcal{C})}^*(-, G'): E(\mathcal{T}^{\text{op}}) \rightarrow \text{Gr}(K)$  for all  $G'$  in  $\mathcal{F}(\mathcal{T})$ , and that  $(F'(G'))^*$  is given by

$$(F'(G'))^*(S) = \text{Hom}_{E(\mathcal{T})^{\text{op}}}(F'(G'), \text{Ext}_{\text{Mod}(\mathcal{C})}^*(-, S))$$

for  $S$  in  $E(\mathcal{T})$ . For  $G' = S'$  in  $\mathcal{T}$  we have that

$$\begin{aligned} (F'(S'))^*(S) &= \text{Hom}_{E(\mathcal{T})^{\text{op}}}(\text{Ext}_{\text{Mod}(\mathcal{C})}^*(-, S'), \text{Ext}_{\text{Mod}(\mathcal{C})}^*(-, S)) \\ &\simeq \text{Ext}_{\text{Mod}(\mathcal{C})}^*(S', S) \\ &= F(S')(S) \end{aligned}$$

Then using similar arguments as for the algebra situation, one can show that  $(-)^*F' \simeq F: \mathcal{F}(\mathcal{T}) \rightarrow \text{gr}E(\mathcal{T})$ . The claim in (c) follows from this.  $\square$

The above result has the following immediate consequence.

**Corollary 6.3.** *Let  $\Lambda$  be a non-semisimple finite dimensional algebra over a field  $k$ , and let  $M$  be an  $n$ -cluster tilting  $\Lambda$ -module. Denote by  $\Gamma$  the endomorphism ring  $\text{End}_\Lambda(M)$  of  $M$ . Let  $T = \Gamma/\mathfrak{r}_\Gamma$ . Then  $\text{Ext}_\Gamma^*(T, T)$  is a finite dimensional Frobenius algebra.*

Theorem 6.2 shows that for any generalized Artin-Schelter regular  $K$ -category  $\mathcal{C}$  with duality and any full subcategory  $\mathcal{T}$  of simple  $\mathcal{C}$ -modules of maximal projective dimension  $n$  such that  $D\text{tr}_\mathcal{C}^2$  induces a bijection on the simple  $\mathcal{C}$ -modules in  $\mathcal{T}$ , the associated Ext-category  $E(\mathcal{T})$  is Frobenius. Here  $\mathcal{C}$  need not to be Koszul. In the Koszul case the converse is also true as we prove next.

**Theorem 6.4.** *Let  $\mathcal{C}$  be a Koszul  $K$ -category, such that  $\text{gr}E(\mathcal{S}(\mathcal{C}))$  is indecomposable for the Ext-category  $E(\mathcal{S}(\mathcal{C}))$  of the full subcategory  $\mathcal{S}(\mathcal{C})$  of  $\mathcal{C}$ -modules generated by the simple objects. Assume that  $\text{gr}E(\mathcal{S}(\mathcal{C}))$  is Frobenius and a finite length category with Loewy length  $n$ . Then  $\mathcal{C}$  is generalized Artin-Schelter regular of dimension  $n$  and all simple objects have the same projective dimension.*

*Proof.* The proof will be the same as in the algebra case given in [21], it will rely on the use of Koszul duality. We give it here for completeness.

Let  $\mathcal{D}$  be an indecomposable Frobenius Koszul  $K$ -category which is a finite length category. First we prove that all indecomposable projective objects in  $\text{gr}(\mathcal{D})$  have the same Loewy length. Let  $P = \text{Hom}_\mathcal{D}(-, C)$  for an indecomposable object  $C$  in  $\mathcal{D}$  of maximal Loewy length. Let  $Q = \text{Hom}_\mathcal{D}(-, B)$  be an indecomposable projective in  $\text{gr}(\mathcal{D})$  not isomorphic to  $P$  with  $Q/\text{rad } Q$  as a direct summand of  $\text{Hom}_\mathcal{D}(-, C)_1$ . By Yoneda's Lemma there is a morphism  $\tilde{f}: Q \rightarrow \text{rad } P$  induced by a morphism  $f: B \rightarrow C$  in  $\text{Hom}_\mathcal{D}(B, C)_1$ . Since  $P$  and  $Q$  both are indecomposable projective injective objects, we infer that the socle of  $Q$  is contained in  $\text{Ker } \tilde{f}$ . It follows that  $\text{LL}(Q) > \text{LL}(\text{rad } P) = \text{LL}(P) - 1$ , where  $\text{LL}(H)$  denotes the Loewy length of a functor  $H$ . Since  $\text{LL}(P)$  is maximal,  $\text{LL}(Q) = \text{LL}(P)$ .

Let  $Q = \text{Hom}_\mathcal{D}(-, B)$  be an indecomposable summand of the injective envelope of  $P/\text{soc } P$ . As above there is a morphism  $f: C \rightarrow B$  in  $\text{Hom}_\mathcal{D}(C, B)_1$ . Similarly as above using the maximality of  $\text{LL}(P)$  we conclude that  $\text{LL}(Q) = \text{LL}(P)$ . We obtain by induction that for all indecomposable objects  $B$ , which has a chain of non-zero maps (in either directions) to  $C$ , the projective objects  $\text{Hom}_\mathcal{D}(-, B)$  has the same Loewy length as  $\text{Hom}_\mathcal{D}(-, C)$ . Since  $\mathcal{D}$  is indecomposable, all (indecomposable) projective objects have the same Loewy length.

Now, since  $\mathcal{D} = \mathcal{E}(\mathcal{S}(\mathcal{C}))$  is assumed to be Frobenius and  $\mathcal{C}$  is a Koszul  $K$ -category, it follows by the proof of [20, Theorem 2.3 (e), (f)] that all the simples in  $\mathcal{C}$  have projective dimension  $n$ . By [20, Theorem 1.18] the category  $\mathcal{C}$  has global dimension  $n$ .

Let  $C$  be an indecomposable object in  $\mathcal{C}$ , and let  $S_C = \text{Hom}_\mathcal{C}(-, C)/\text{rad}(-, C)$ . We want to show that  $\text{Ext}_{\text{Mod}(\mathcal{C})}^i(S_C, \text{Hom}_\mathcal{C}(-, X)) = (0)$  for  $i \neq n$  for all objects  $X$  in  $\mathcal{C}$ . Let

$$0 \rightarrow P_n[-n] \rightarrow P_{n-1}[-n+1] \rightarrow \cdots \rightarrow P_1[-1] \rightarrow P_0 \rightarrow S_C \rightarrow 0$$

be a minimal graded projective resolution of  $S_C$ .



Let  $Q = \text{Hom}_{\mathcal{C}}(-, X)$  be an indecomposable projective in  $\text{gr}(\mathcal{C})$ . Recall that we considered the functor  $E: \text{gr}(\mathcal{C}) \rightarrow \text{Gr}E(\mathcal{S}(\mathcal{C}))$  given by

$$E(H) = \text{Ext}_{\text{Mod}(\mathcal{C})}^*(H, -): E(\mathcal{S}(\mathcal{C})) \rightarrow \text{Gr}(K)$$

induces a duality between the subcategory of linear functors  $\mathcal{L}(\mathcal{C})$  in  $\text{Mod}(\mathcal{C})$ , and the subcategory of linear functors  $\mathcal{D}$  in  $\text{Mod}(\mathcal{D})$ .

An element of  $\text{Ext}_{\text{Mod}(\mathcal{C})}^i(S_C, \text{Hom}_{\mathcal{C}}(-, X))$  is represented by a degree zero map  $h: \Omega^i(S_C) \rightarrow Q[-m]$ , where  $Q = \text{Hom}_{\mathcal{C}}(-, X)$ . Since  $E(\text{rad}^i Q[i]) \simeq \Omega^i E(Q)[i]$  and  $E(Q)$  is simple,  $E(\text{rad}^i Q)$  is indecomposable and therefore  $\text{rad}^i Q$  is indecomposable. Similarly,  $E(\Omega^i(S_C)[i]) \simeq \text{rad}^i E(S_C)[i]$  is a subfunctor of an indecomposable projective injective functor of finite length, hence with simple socle. It follows that any subfunctor of  $E(S_C)$  is indecomposable, in particular,  $\text{rad}^i E(S_C)$  is indecomposable, then  $\Omega^i(S_C)$  is indecomposable. From these observations and the fact that projective dimension of  $S_C$  is  $n$ , it follows the map  $h$  is not an epimorphism, hence  $m < i$  and  $\text{Im } h \subset \text{rad}^{i-1-m} Q[-m]$ . Consider the following push out

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega^i(S_C) & \longrightarrow & P_{i-1}[-i+1] & \longrightarrow & \Omega^{i-1}(S_C) \longrightarrow 0 \\ & & \downarrow h & & \downarrow & & \parallel \\ 0 & \longrightarrow & \text{rad}^{i-1-m} Q[-m] & \longrightarrow & Z & \longrightarrow & \Omega^{i-1}(S_C) \longrightarrow 0 \end{array}$$

Denote the lower exact sequence in the above diagram by  $\theta$ . The end terms in  $\theta$  are linear functors generated in degree  $i-1$ , therefore  $Z$  is a linear functor generated in degree  $i-1$ . Hence the exact sequence,  $\theta[i-1]$ ,

$$0 \rightarrow \text{rad}^{i-1-m} Q[i-1-m] \rightarrow Z[i-1] \rightarrow \Omega^{i-1}(S_C)[i-1] \rightarrow 0$$

is a sequence of linear functors. Applying Koszul duality we obtain an exact sequence linear functors

$$0 \rightarrow E(\Omega^{i-1}(S_C)[i-1]) \rightarrow E(Z[i-1]) \rightarrow E(\text{rad}^{i-1-m} Q[i-1-m]) \rightarrow 0$$

in  $\text{Mod}(\mathcal{D})$ , which is isomorphic to the following sequence

$$\theta': 0 \rightarrow \text{rad}^{i-1} E(S_C)[i-1] \xrightarrow{g} E(Z[i-1]) \rightarrow E(\text{rad}^{i-1-m} Q[i-1-m]) \rightarrow 0$$

There is an inclusion map  $j: \text{rad}^{i-1} E(S_C)[i-1] \rightarrow E(S_C)[i-1]$  and  $E(S_C)[i-1]$  it is projective injective, it follows there exists a map  $f: E(Z[i-1]) \rightarrow E(S_C)[i-1]$  such that  $fg = j$  the image of  $f$  is contained in  $\text{rad}^{i-1} E((S_C))[i-1]$ . Therefore the exact sequence  $\theta'$  splits. By Koszul duality the exact sequence  $\theta$  also splits. We have proved the map  $h: \Omega^i(S_C) \rightarrow Q[-m]$  extends to  $P_{i-1}[-i+1]$ , and the corresponding extension splits.

If  $\text{Hom}(S_C, Q) \neq (0)$ , then there is an integer  $k$  such that  $S_C$  is a summand of  $\text{rad}^k Q$ , but we saw above that  $\text{rad}^k Q$  is indecomposable. This implies  $\text{rad}^k Q = S$  and  $Q$  is of finite length, which would imply  $E(Q)$  has finite projective dimension contradicting  $\mathcal{D}$  is Frobenius.

Dualizing with  $(-)^*$  the exact sequence

$$0 \rightarrow P_n[-n] \rightarrow P_{n-1}[-n+1] \rightarrow \cdots \rightarrow P_1[-1] \rightarrow P_0 \rightarrow S_C \rightarrow 0$$

and shifting by  $n$ , we obtain the exact sequence

$$0 \rightarrow P_0^*[-n] \rightarrow P_1^*[-n+1] \rightarrow \cdots \rightarrow P_{n-1}^*[-1] \rightarrow P_n^* \rightarrow M \rightarrow 0$$

Hence, since  $P_n$  is indecomposable, the functor  $M$  is linear of projective dimension  $n$  and indecomposable.

Since the opposite category is Koszul, we can apply Koszul duality to  $M$  to get an indecomposable functor  $E(M)$  in  $\text{Gr}(\mathcal{D})$  of Loewy length  $n$ , but the only indecomposable functors of length  $n$  in  $\text{Gr}(\mathcal{D}^{\text{op}})$  are the projective. It follows  $M$  is simple. We have proved  $\text{Ext}_{\text{Mod}(\mathcal{C})}^n(S_C, \text{Hom}_{\mathcal{C}}(-, X))$  is simple.  $\square$

Let us now briefly return to the polynomial ring  $\Lambda = K[x_1, x_2, \dots, x_n]$  in  $n$  indeterminates  $x_1, x_2, \dots, x_n$  over an algebraically closed field  $K$  discussed in Section 3. Proposition 2.2 (b) in particular says that the Yoneda algebra  $\prod_{i \geq 0} \text{Ext}_{\Lambda}^i(K, K)$  is Frobenius. This is well-known and it is in fact isomorphic to the exterior algebra on  $K^n$ . However we can consider  $\Lambda$  as an ungraded  $K$ -algebra, and we have seen that  $\Lambda$  is a generalized Artin-Schelter regular  $K$ -algebra. If  $\mathcal{C}$  is the category with one object  $*$  with endomorphism ring  $\Lambda$ , then the category of simple objects  $E(\mathcal{S})$  in  $\text{Mod}(\mathcal{C})$  is the category of simple  $\Lambda$ -modules  $S_{\bar{a}} = \Lambda/\mathfrak{m}_{\bar{a}}$  with  $\mathfrak{m}_{\bar{a}} = (x_1 - a_1, x_2 - a_2, \dots, x_n - a_n)$  and homomorphisms

$$\text{Hom}_{E(\mathcal{S})}(S_{\bar{a}}, S_{\bar{b}}) = \prod_{i \geq 0} \text{Ext}_{\Lambda}^i(S_{\bar{a}}, S_{\bar{b}}).$$

Each  $\text{Ext}_{\Lambda}^i(S_{\bar{a}}, S_{\bar{b}})$  is a module over  $\Lambda$ , where the action of  $\Lambda$  can be taken through  $S_{\bar{a}}$  or  $S_{\bar{b}}$ . Hence  $\text{Ext}_{\Lambda}^i(S_{\bar{a}}, S_{\bar{b}})$  is annihilated by  $\mathfrak{m}_{\bar{a}} + \mathfrak{m}_{\bar{b}} = \Lambda$  if  $\bar{a} \neq \bar{b}$ . Therefore  $\text{Gr}(E(\mathcal{S}))$  splits in a product of categories with each one isomorphic to the category of modules over  $\prod_{i \geq 0} \text{Ext}_{\Lambda}^i(S_{\bar{a}}, S_{\bar{a}})$ . Since  $S_{\bar{a}} \simeq K^{\sigma_{\bar{a}}}$ , it follows that each of these algebras are isomorphic to the exterior algebra.

## 7. GENERALIZED ARTIN-SCHELTER REGULAR CATEGORIES OF GLOBAL DIMENSION 2

The main example providing the motivation for this work is the category  $\text{Mod}(\text{mod } \Lambda)$  of all additive functors from  $(\text{mod } \Lambda)^{\text{op}}$  to  $\text{Mod } K$  for a finite dimensional  $K$ -algebra  $\Lambda$ . However, in our definition of a generalized Artin-Schelter regular  $K$ -category  $\mathcal{C}$ , we require that the category is Noetherian, that is,  $\text{Mod } \mathcal{C}$  is Noetherian. It is shown by Auslander in [3, Theorem 3.12] that  $\text{Mod}(\text{mod } \Lambda)$  is Noetherian if and only if  $\Lambda$  is of finite representation type. However, if we consider the associated graded category,  $\mathcal{A}_{\text{gr}}(\text{mod } \Lambda)$ , of  $\text{mod } \Lambda$ , that is, the objects are the same as for  $\text{mod } \Lambda$  and the morphism is given by

$$\text{Hom}_{\mathcal{A}_{\text{gr}}(\text{mod } \Lambda)}(A, B) = \bigoplus_{i \geq 0} \text{rad}_{\Lambda}^i(A, B) / \text{rad}_{\Lambda}^{i+1}(A, B).$$

Then consider the graded  $K$ -category  $\text{Gr}(\mathcal{A}_{\text{gr}}(\text{mod } \Lambda))$  of graded functors from  $(\mathcal{A}_{\text{gr}}(\text{mod } \Lambda))^{\text{op}} \rightarrow \text{Gr}(K)$  (see [20]). Then by [9] the category  $\mathcal{C} = \text{Gr}(\mathcal{A}_{\text{gr}}(\text{mod } \Lambda))$  is a generalized Artin-Schelter regular  $K$ -category of dimension 2. Furthermore,  $\text{Gr}(\mathcal{A}_{\text{gr}}(\text{mod } \Lambda))$  gives rise to finite length Frobenius categories through naturally associated Ext-categories. This is why we discuss Krull-Schmidt categories  $\mathcal{C}$  where all finitely generated projective objects in  $\text{Mod}(\mathcal{C})$  has finite length and generalized Artin-Schelter regular  $K$ -categories of dimension 2 in this section. We show that an indecomposable positively graded locally finite Krull-Schmidt  $K$ -category generated in degrees 0 and 1 with finite presented simple objects and finitely generated projective objects in  $\text{Mod}(\mathcal{C})$  have finite length, then the number of isomorphism classes of indecomposable objects in  $\mathcal{C}$  is countable. In addition, a generalized Artin-Schelter regular positively graded locally finite  $K$ -category generated in degrees 0 and 1 of dimension 2 is coherent and  $\text{gr}(\mathcal{C})$  is abelian.

We start by showing that the number of isomorphism classes of indecomposable objects in a positively graded locally finite  $K$ -category generated in degrees 0 and 1 with finitely presented simple objects in  $\text{Mod}(\mathcal{C})$  is countable.

**Lemma 7.1.** *Let  $\mathcal{C}$  be a positively graded locally finite  $K$ -category generated in degrees 0 and 1 such that the graded simple objects  $S_C = \text{Hom}_{\mathcal{C}}(-, C)_0$  are finitely presented. Assume that  $\mathcal{C}$  is indecomposable and that each projective object has finite length. Then the isomorphism classes of indecomposable objects in  $\mathcal{C}$  is countable.*

*Proof.* Let  $C$  be an object in  $\mathcal{C}$  and denote by  $\langle C \rangle$  the full subcategory of  $\mathcal{C}$  consisting of all objects  $X$  such that there is a chain of morphisms in  $\mathcal{C}$

$$X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \cdots \cdots \cdots X_{n-1} \xrightarrow{f_{n-1}} X_n = C,$$

where  $f_i$  is either in  $\text{Hom}_{\mathcal{C}}(X_{i-1}, X_i)_1$  or  $\text{Hom}_{\mathcal{C}}(X_i, X_{i-1})_1$ .

The fact that simple objects are finitely presented means that there are only a finite number of objects  $X$  connected to an object  $Y$  with a map in degree 1.

Since each projective object has finite length, and  $\mathcal{C}$  is generated in degrees 0 and 1, then each homogeneous morphism  $f: X \rightarrow C$  in  $\mathcal{C}$  with a positive degree  $l$  is a sum of composition of maps  $f_1 f_2 \cdots f_l$  with  $f_i$  in degree 1. Then  $\langle C \rangle$  is countable. If  $D$  is in  $\mathcal{C} \setminus \langle C \rangle$ , then by the definition of  $\langle C \rangle$  there is no morphism from  $D$  to any object of  $\langle C \rangle$  and from any object of  $\langle C \rangle$  to  $D$ . This contradicts the fact that  $\mathcal{C}$  is connected, so that we have  $\mathcal{C} = \langle C \rangle$  and  $\mathcal{C}$  is countable.  $\square$

Note that when  $\mathcal{C}$  is a generalized Artin-Schelter regular Koszul  $K$ -category of dimension  $n$  with duality  $D$  and  $\mathcal{T}$  is a set of simple  $\mathcal{C}$ -modules in  $\mathcal{H}_{\mathcal{C}}^n$  permuted by  $\text{Dtr}_{\mathcal{C}}^n$ , then the Ext-category  $E(\mathcal{T})$  satisfies the assumptions in Lemma 7.1.

We end this section and this paper with showing that generalized Artin-Schelter regular  $K$ -categories of dimension 2 generated in degrees 0 and 1 are coherent. To this end recall the definition of the functor  $t^n: \text{Mod}(\mathcal{C}) \rightarrow \text{Mod}(\mathcal{C})$  preceding Lemma 5.3. Since we are discussing the global dimension 2 case in this section, we are considering the functor  $t^2$ .

**Theorem 7.2.** *Let  $\mathcal{C}$  be a generalized Artin-Schelter regular positively graded locally finite  $K$ -category generated in degrees 0 and 1 of global dimension 2. Then the following statements are true.*

- (a) *If  $F$  is a finitely generated functor, then  $t^2(F)$  is of finite length.*
- (b) *The category  $\mathcal{C}$  is coherent.*

*Proof.* It was proved in Lemma 5.3 that  $t^n$  is a radical. Furthermore, for any (finitely presented) functor  $F$ , the functor  $F/t^2(F)$  is  $\mathcal{H}_{\mathcal{C}}^2$ -torsion free and therefore has a projective dimension at most 1 by Lemma 5.4.

Consider the commutative exact diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \Omega F & \xrightarrow{j} & \Omega(F/t^2 F) & \longrightarrow & t^2 F \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & P & \xlongequal{\quad} & P & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & t^2(F) & \longrightarrow & F & \longrightarrow & F/t^2(F) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 
 \end{array}$$

where  $\Omega(F/t^2 F) \simeq \oplus_{i \in I} Q_i$  and each  $Q_i$  projective. Since  $\Omega F$  is finitely generated, there exists a finite subset  $J \subset I$  such that  $j(\Omega F) \subset \oplus_{i \in J} Q_i$ .

We have the following commutative exact diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \Omega F & \longrightarrow & \oplus_{i \in J} Q_i & \longrightarrow & Z \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \Omega F & \longrightarrow & \oplus_{i \in I} Q_i & \longrightarrow & t^2(F) \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & \oplus_{I \setminus J} Q_i & \xlongequal{\quad} & \oplus_{I \setminus J} Q_i \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

each  $Q_i$  with  $i$  in  $I \setminus J$  is projective and  $\mathcal{H}_C^2$ -torsion, therefore it has a simple subobject  $S$  from  $\mathcal{H}_C^2$  and  $L = Q_i/S$  has projective dimension 3, a contradiction. It follows that  $I$  is finite and that  $t^2 F$  has finite length. By the Horseshoe Lemma,  $F$  has a projective resolution consisting of finitely generated projective. It follows  $\mathcal{C}$  is coherent.  $\square$

This has the following immediate consequence.

**Corollary 7.3.** *Let  $\mathcal{C}$  be a generalized Artin-Schelter regular positively graded locally finite  $K$ -category generated in degrees 0 and 1 of global dimension 2. Then the category of finitely presented functors in  $\text{Gr}(\mathcal{C})$  is abelian.*

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